The Recovery Theorem

An analysis for unbounded diffusions

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Abstract

Some say that finance has its equivalent to the dark matter cosmologists posit to explain the behaviour of their models for the universe when observables seem insufficient. *"The dark matter of finance is the very low probability of a catastrophic event and the impact that changes in the perceived probability can have on asset prices."*[1, p.618] Historically the average return on equity has far exceeded the average return on short-term virtually default-free debt. General equilibrium models struggle to rationalize this finding. The level of risk aversion needed to explain this difference is not consistent with other branches of economics (see [2]). This conundrum was coined *The Equity Premium Puzzle* by Mehra and Prescott.[2] Within the general equilibrium framework, Thomas A. Rietz proposed the following solution to this headache: *"The effects of possible, though unlikely, market crashes,..., allows to explain the high equity risk premia and low risk-free returns."*[3, p.117] Further, Rietz marks that: *"To the extent that equity returns have been high with no crashes, equity owners have been compensated for the crashes that happened not to occur."*[3, p.118] Historical time series of returns may be a bit too silent about the possibility of extreme adverse events and thus give flawed insight into the risk associated to equity markets. Since historical data are less helpful than we would like, turning to the current *forecast* may give us valuable insight into market sediment. From Ross we have: *"When we extract the risk-neutral probabilities from the price of options on the S&P 500, we find the risk-neutral probability of, for example, a 25% drop in a month, to be higher than the probability calculated from historical stock returns."*[1, p.618] When inspecting risk-neutral probabilities, we have the problem of separating between the predicted *natural* probability distribution, and thus the beliefs about, for example, severe drops in the market, and the risk premium. Harvesting the information embedded in option prices is the goal of this thesis.

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Contents

1 Introduction

”What is mankind’s greatest invention? Ask people this question and they are likely to pick familiar technologies such as printing or electricity. They are unlikely to suggest an innovation that is just as significant; the financial contract. Widely disliked and often considered grubby, it has nonetheless played an indispensable role in human development for at least 7,000 years.”[4]

In a simplifying manner, finance does just two things. Firstly, it gives the possibility of moving surplus into the future and giving borrowers access to future earnings now. Secondly, financial contracts can provide a safety net, insuring against flood, fires or illness. These two services help making an uncertain world more predictable.

Financial contracts with an insurance purpose have been used for a long time. Aristotle describes an option contract in his work *Politics.*[5] Another often quoted ancient reference to a transaction with an option feature can be found in the Bible, Genesis 29, where Laban offers Jacob an option to marry his youngest daughter Rachel in exchange for seven years of labour. [5]
During the medieval era as trade expanded and the importance of urban centres rose, contracting became essential to urban merchants contracting with agricultural producers for crops prior to harvest. Today, the option and future market is vast; *The Economist* has reported as of June 2011, the over-the-counter (OTC) derivatives market amounted to approximately $700 trillion.[6] Due to the size and widespread use of these financial instruments, pricing is well developed. But what is different with pricing of options and futures compared with pricing goods and services in more plain vanilla markets?

Supply and demand determine the price for most goods and services. Prices and resulting volumes adjust up or down such that quantity supplied equals quantity demanded. The market clears. There is something fundamentally different with option and future markets. These kinds of financial instruments are agreements specifying trade in the uncertain future, making the parties involved legally committed to fulfil their part of the contract. Since these obligations stretch into the future, it is clear that whether or not such a contract is profitable for the parties involved depends on how the future unveils. If we today commit to buy one barrel of crude oil in one year from now for a price specified today, we do not know if this price will be above or below the market price at the time of exercise of this contract. As we know, price of crude oil in one year is impossible to predict with certainty. What we can do instead is to *indicate probable* values of this price and base our contract on these beliefs.

From this line of argument, it becomes clear that option and future prices must be based on some beliefs about the future. For firms issuing such contracts, their existence depends on forming opinions about economic important sizes in the future. Since the size of these markets is considerable and the firms issuing such contracts in general make a lot of money, their methodology seems to be well-developed. Many consider stock indexes as a bell-wether for the economy, and in the US, The National Bureau of Economic Research has classified common stock as leading for the business cycles.[7] Bhupinder Bahra, Bank of England, writes:

"Many monetary authorities routinely use the information that is embedded in financial asset prices to help in formulating and implementing monetary policy. In this context, derivative markets provide them with a rich source of information for gauging market sentiment; due to their forward-looking nature, futures and options prices efficiently encapsulate market perceptions about underlying asset prices in the future."[8, p.7]

In 1973, Fisher Black and Myron Scholes published an entirely new and innovative analysis of option valuation.[9] In an idealized setting they showed that an owner of a call option (giving the right but not the obligation to buy some asset in the future for an amount specified today), could simultaneously buy and sell the underlying stock in such a way as to exactly match the return of the option. Having available two investment opportunities with exactly the same return effectively eliminates all risk, by allowing an investor to buy one while selling the other. The main lesson from this argument was that since writing an option potentially carries no risk, its return must be the same as for other riskless investments in the economy. Otherwise, limitless profit opportunities bearing no risk would arise. This lead to the now famous Black and Scholes partial differential equation(PDE). The solution of this PDE and hence the value of an option is the discounted expected present value taken under a different probability measure; the *risk-neutral* probability measure. *"In other words, to find the option premium we need to operate in a probability universe for which the stock price has slightly different properties than in the real-world."*[10, p.5]

With this in mind, Stephen Ross' recent paper "The Recovery Theorem" is remarkable.[1] Even though options can be valued without knowing the expected return, Ross uses option prices to infer not only the average natural return, but also the entire natural return distribution. Risk-neutral returns are natural returns that have been adjusted for risk. In a universe where investors do not care about risk, all assets must yield the same expected return. This means that in the risk-neutral universe, the expected return on all assets is the risk-free rate, and this is obtained by adjusting the natural return with some risk premium. The risk premium depends both on risk and the market's risk aversion. Therefore, if we are going to use the risk-neutral probabilities inferred from option prices to estimate natural probabilities we have to know the risk adjustment. Under some assumptions Ross is able to do exactly this; determine the market's return and the risk aversion from option prices.

My interest for Ross' Recovery Theorem awoke during NBIMs summer school 2014. Our teacher, professor William E. Goetzmann from Yale (who also has written several papers with Steve Ross), told us about the remarkable ”Ross Recovery” and how interesting it would be to understand how Ross had obtained this fascinating result. I am not the only one who has become curious about Ross' startling result. Many articles have been written on this topic in the last two years, either trying to do this empirically, tweaking a bit on Ross' setup or trying to prove the results in a more general setting. It will be easier for me to explain what I have intended to investigate in my master thesis if I first highlight some of the methodology already used and discuss possible drawbacks with the existing literature. There are two main approaches to recovery:

Ross chooses to model the economy in discrete-time and assumes finite states of the world. Typical states would be ”good,” ”normal,” ”bad” or ”crises” where each of these states have different economical properties, say for example return on some investment opportunity. He further assumes that a typical, or representative, agent has the possibility of buying so called *state-price securities*. A *state-price security*[1] is a contract that agrees to pay one unit of a numeraire (a currency or a commodity) if a particular state occurs at a particular time in the future and pays zero numeraire in all the other states. The agent is then faced with the problem of investing in the state-price securities in an optimal way maximizing expected utility.
The main restriction of Ross' approach is that the forecast is only valid for a stock market index, taken as a proxy for the holdings of the representative agent. This means that if some asset or some index could not possibly represent the entire holdings of the typical agent, then Ross' model does not provide a forecast. By studying Ross' paper, we understand that Ross' conclusions rely on the restrictions on the preferences of the representative agent. Ross is counting on the Von-Neumann-Morgenstern axioms[11] that lead to the conclusion that all individuals behave as if they maximize utility. These axioms have been the subject of much debate. As an example, *The Allais paradox*[12] is a choice problem designed to show an inconsistency of actual observed choices with the predictions of expected utility theory. Ross also uses time separable and state independent utility functions. As Carr and Yu highlight in their article, this excludes satiation effects and habit formation. *”One's utility from consuming sushi for dinner is independent of whether one had sushi for lunch.”*[13, p.41]

Contrary, Carr and Yu[13] model the dynamics of some creation called the *numeraire portfolio*, and the goal of their analysis is to impose structure on the real-world dynamics of the numeraire portfolio in order to identify the random variable linking the risk-neutral probability measure $\mathbb{Q}$ with the real-world probability measure $\mathbb{P}$. A strength with this methodology is that their forecast is valid for the underlying of any derivative security, even if it is unimportant or not traded. Their model is only valid when the driving, underlying

[1] also called an Arrow-Debreu security

stochastic process lives on a bounded domain.

As the great economist Paul Samuelson famously said:"The stock market has forecast nine of the last five recessions."[13, p.39] It would be interesting to investigate whether the option market can produce a better record than its underlying stock market.
Since it is common in mathematical finance to model the economy by unbounded stochastic processes, my goal is to investigate Carr and Yu's model carefully and see if I can find some way of extending their analysis to unbounded diffusions. Hence it would be easier analysing this remarkable theory on real-world data. For example, when modeling stock prices, Geometrical Brownian Motion(GBM) is often considered and this process is not bounded.[2] Further, when modeling the interest rate market, some modifications of the Ornstein-Uhlenbeck process are often considered, as the Vasicek, CIR or the Hull-White model and these processes are not bounded.

2 The Model

Peter Carr is the Global Head of the Market Modeling division at Morgan Stanley, has 15 years of experience from the derivative industry, has been a finance professor for 8 years at Cornell and holds a Ph.D. from UCLA. Jiming Yu is Vice President at Morgan Stanley with over 7 years of experience from the banking industry and has a Ph.D. in Electrical Engineering from Princeton. In other words, they are lightyears ahead of me when it comes to modeling. Accordingly, some elaboration is needed. Understanding their model is the first goal. Secondly, in their model recovery is derived for bounded diffusions. My goal is to apply recovery on real life data and using unbounded diffusions to model financial markets is the most common approach. Consequently, I will try to extend their result to unbounded diffusions, or at least investigate recovery for some diffusions of interest.

2.1 Preliminaries on stochastic analysis

We will use continuous time stochastic calculus as a mathematical tool for financial modeling. A central result in this theory is the so-called *Fundamental Theorem of Asset pricing*. A simple version of this theorem states that:

Theorem 2.1 (Fundamental Theorem of Asset Pricing [15])**.** *The market $\mathcal{M}$ is free of arbitrage if and only if there exist a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the discounted d-dimensional asset price process $\{\tilde{S}_t\}_{t\geq 0}$ is a $\mathbb{Q}$-martingal.*

[2]Under some parameter values this process possesses the property that the process is drifting ever closer to zero, while simultaneously, the expected value is continuously increasing and approaches ∞![14, p.425]

Two probability measures are equivalent if they agree on which events have probability zero. To understand what this theorem says, and to facilitate the reading, we will now explain some of the concepts used.

An important mathematical concept in the world of arbitrage-free pricing is a *filtered probability space.* Firstly, a *probability space*, $(\Omega, \mathcal{F}, \mathbb{P})$, is a mathematical construction used to model a real-world process. The interpretation of this space as an experiment can help intuition. The generic experiment result is denoted by $\omega \in \Omega$, where Ω represents the set of all possible outcomes of the random experiment. The σ-algebra $\mathcal{F}$ represents the sets of events $A \subset \Omega$ with a certain *property.*[3] $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a function assigning values to some event $A \in \mathcal{F}$.

Secondly, the temporal feature of a stochastic process suggests a flow of time, in which at every moment $t \geq 0$ we can talk about a *past, present* and *future.* A mathematical creation useful to handle this, is a *filtration.*

Definition 2.1 (Filtration[16])**.** *A filtration on* $(\Omega, \mathcal{F})$ *is a family* $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ *of* σ*-algebras* $\mathcal{F}_t \subset \mathcal{F}$ *such that*

$$0 \leq s < t \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t$$

Given a stochastic process X_t, a natural choice of filtration is that generated by the process itself.

$$\mathcal{F}_t^X := \sigma(X_s; 0 \leq s \leq t)$$

Subsequently we let our filtration be generated by our stochastic process of interest and suppress the "X-notation."

The σ-algebra $\mathcal{F}_t$ represents the information available up to time t. Due to the inclusion, we have that information increases in time, never exceeding the whole set of events $\mathcal{F}$. If the experiment result is ω and $\omega \in A \in \mathcal{F}$, we say that the event A occurred. We interpret $A \in \mathcal{F}_t$ to mean that by time t, an observer of X_t knows whether or not A has occurred. If $\omega \in A \in \mathcal{F}_t$, we say that the event occurred at time before or equal to t. A filtration allows us to distinguish between events that are known to us at time t given the information $\mathcal{F}_t$ from those events which still have to be seen as random at that timepoint.

[3]Let Ω be a set. A nonempty collection $\mathcal{F}$ of subsets of Ω is called a σ*-algebra* if the following conditions are satisfied:

(i) $\varnothing \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2, ... \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Øksendal begins his book by explaining how allowing some randomness in the coefficients of a differential equation often will give a more realistic mathematical model of some phenomena.[16] To illustrate, consider a simple model for evolution of Gross Domestic Product (GDP) over time. Let x_t be GDP at time t. The simplest model for evolution in GDP is obtained by assuming change, $\frac{dx_t}{dt}$, is proportional to the current level of GDP. This can be translated into the following differential equation:

$dx_t = gx_t dt, \qquad x_0$; initial value, and g a real constant.

If we do not exactly know the level of GDP at the starting time, meaning x_0 is a random variable, the solution of the equation above will inherit the uncertanty and be a random variable itselves.

An economy is a complicated structure. If we are interested in describing the evolution over time of some economic phenomena, it might not be realistic to claim that we with certainty can pin down forces driving this process. In analogy to the setting above, we can assume our knowledge of g is perturbed by some randomness. We can model this by some stochastic process $\{W_t(\omega), t \geq 0\}$. Extending our differential equation from above we get:

$dX_t(\omega) = \Big(gdt + dW_t(\omega)\Big)X_t(\omega), X_0(\omega).$

A natural question is: "what properties should the process $\{W_t(\omega), t \geq 0\}$ have?" In many situations arising in engineering or when studying nature, one is led to assume $t_1 \neq t_2$ implies W_{t_1} and W_{t_2} are independent, the distribution of $\{W_t\}_{t\geq 0}$ does not depend on t and the expected perturbation is 0. It turns out that if we look at *increments*, the only process satisfying these conditions with continuous paths is the Brownian motion, B_t. In many situations, allowing discontinuities in the sample paths will create an even more realistic model.

Let $\eta = \Big\{\eta(t); t \geq 0\Big\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that η is a *Lévy process* if:

(i) $\eta(0) = 0$ *almost sure*(*a.s.*[4])

(ii) η has *independent* and *stationary* increments

(iii) η is *stochastically continuous*, i.e. for all $\epsilon > 0$ and for all $s \geq 0$
$\lim_{t\to s} \mathbb{P}\Big(|\eta(t) - \eta(s)| > \epsilon\Big) = 0$

Further, if η is a Lévy process, then there exists a drift coefficient, $b \in \mathbb{R}^d$, a Brownian motion, $B_{\mathbf{A}}$, with covariance matrix $\mathbf{A}$ and an independent Poisson random measure N on $\mathbb{R}^+ \times \Big(\mathbb{R}^d - \{0\}\Big)$ such that, for each $t \geq 0$

$$\eta(t) = bt + B_{\mathbf{A}}(t) + \textstyle\int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx)$$

[4]Let F denote a possible event and let $F \in \mathcal{F}$. We say that F happens almost surely if $\mathbb{P}(F) = 1$.

This result is called the *Lévy-Itô* decomposition theorem and it says that *any* Lévy process $\eta(t)$ can be decomposed into a Brownian motion and a *pure and compensated jump part* consisting of Poisson processes with different jump sizes. The jump of η_t at time t is defined by $\Delta\eta_t := \eta_t - \eta_{t^-}$. $N(t,U) = N(t,U,\omega) = \sum_{0<s\leq t} \chi_U(\Delta\eta_s)$ is called the *jump measure* of η. In other words, $N(t,U)$ is the number of jumps of size $\Delta\eta_s \in U$ which occur before or at time t. $\tilde{N}(t,U)$ is just the jump measure subtracted some normalizing constant.[17]

The properties of Lévy processes are considered reasonable in models of financial markets. Prior evolution of prices should not help us predict future prices.
Since Brownian motion is so important in the study of Lévy processes, yet keeps the analysis simpler and more tractable than in the general case, we will restrict ourselves to only consider Brownian motion. Brownian motion is a Lévy-process where the *increments* are normally distributed with variance rate dependent on time. That is $B_t(\omega) - B_s(\omega) \sim \mathcal{N}(0, t-s)$ for $t \geq s$.

Definition 2.2 (Stochastic differential equation[18]). *A typical stochastic differential equation (SDE) is on the following form:*

$$dX_t = b(t, X_t)dt + a(t, X_t)dB_t \tag{2.1}$$

where $a, b : [0, \infty) \times \Omega \to \mathbb{R}$ are some functions and B_t is a standard Brownian motion. If the functions a, b are bounded by linear growth and *Lipschitz*[5] there exists a unique solution of (2.1). It turns out that the paths $t \to B_t(\omega)$ of Brownian motion are a.s. nowhere differentiable. In fact, the paths have unbounded variation, and hence $\frac{d}{dt}B_t(\omega)$ does not exist. Since $\frac{d}{dt}B_t(\omega)$ does not exist, how should we then interpret the expression in (2.1)? The answer relies on rewriting eq. (2.1) in *integral form*:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t a(s, X_s)dB_s \tag{2.2}$$

But a new question now arises. What does it mean to *integrate with respect to, in our case, Brownian motion*? A priori it is not possible to define it as a Stieltjes integral on the paths, since they have unbounded variation.
To gain insight we can first consider some functions that are not too ill-behaved. Consider an elementary function, $\phi(t,\omega) = \sum_j e_j(\omega)\chi_{(t_j, t_{j+1}]}(t)$, meaning a function that is constant over intervals. For such function it is reasonable to define:

$$\textstyle\int_0^T \phi(s,\omega)dB_s(\omega) := \sum_{j\geq 0} e_j(\omega)[B_{t_{j+1}} - B_{t_j})(\omega).$$

Without further assumption on the function $e_j(\omega)$ we have some difficulties. This is where K. Itô's *choice* is crucial. Itô suggested that the *left end point* should be chosen. We *define* the Itô integral for bounded, elementary functions to be:

[5] We say that a function b is (*globally*) *Lipschitz* if there exists $K > 0$ such that, for all $x, y \in R^n$, $|b(x) - b(y)| \leq K|x - y|$

Definition 2.3 (Itô integral for elementary functions [16])**.**
$\int_0^T \phi(s,\omega)dB_s(\omega) := \sum_{j\geq 0} e_{t_j}(\omega)[B_{t_{j+1}} - B_{t_j})(\omega)$

As Applebaum writes:"*before we analyse this object, we should sit back and gasp at the breathtaking audacity of this prescription.*"[19, p.221] The key point in the definition above, is that for each time interval, $[t_j, t_{j+1}]$, e_{t_j} is adapted to the past filtration $\mathcal{F}_{t_j}$ while $[B_{t_{j+1}} - B_{t_j})$ "sticks into the future" and is independent of $\mathcal{F}_{t_j}$. By conditioning, we are always able to separate the function value and the driving noise, enabling us to to exploit the probabilistic properties of Brownian motion. From this we are able to establish the extremely useful and important *Itô Isometry* giving a surprising expression for an Itô integral in $L^2(\mathbb{P})$, namely:

Lemma 1 (Itô isometry [16])**.** *For a bounded and adapted process θ_t,*

$$\mathrm{E}^{\mathbb{P}}\Big[(\textstyle\int_0^T \theta_t dB_t)^2\Big] = \mathrm{E}^{\mathbb{P}}\Big[\textstyle\int_0^T \theta_t^2 dt\Big].$$

Starting with elementary functions we are able to construct a *class of Itô integrable functions.*

Definition 2.4 ([16])**.** *Let $\mathcal{V}[0,T]$ be the class of functions*

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

such that

(i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}-$ measurable, where $\mathcal{B}$ denotes the Borel-σ algebra on $[0,\infty)$

(ii) f is $\mathcal{F}_t$-adapted

(iii) $\mathrm{E}[\int_0^T f(t,\omega)^2 dt] < \infty$

A stochastic variable $X : \Omega \to \mathbb{R}^n$ is $\mathcal{F}$ measurable if $X^{-1}(U) := \{\omega; X(\omega) \in U\} \in \mathcal{F}$ for all open sets $U \in \mathbb{R}^n$. A process $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{F}_t$-adapted if for each $t \geq 0$ the function $\omega \to f(t,\omega)$ is $\mathcal{F}_t$ measurable. When we fix t we have a stochastic variable. If f is $\mathcal{F}_t$ adapted we should be able to read the value of the stochastic variable $\omega \to f(t,\omega)$ based only on the values the process generating the noise takes up and until time t. An adapted process, or non-anticipating process, is one that cannot "see into the future."

Definition 2.5 (The Itô integral [16])**.** Let $f \in \mathcal{V}(0,T)$. Then the Itô integral of f is defined by

$$\textstyle\int_0^T f(t,\omega)dB_t(\omega) := \lim_{n\to\infty} \int_0^T \phi_n(t,\omega)dB_t(\omega) \qquad \text{(Limit in } L^2(\mathbb{P})\text{)}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$\mathrm{E}^{\mathbb{P}}\Big[\big(f(t,\omega) - \phi_n(t,\omega)\big)^2 dt\Big] \to 0 \text{ as } n \to \infty$$

The fundamental theorem of asset pricing is basically saying that if we we can not find any strategy of investing giving us a sure payoff without taking any risk, then there is some artificial probability measure related to the original probability measure making discounted values of any traded financial object into martingales. *"One cannot win for certain by betting on a martingale"*.[20, p.33]
Understanding what a martingale is will therefore be crucial.

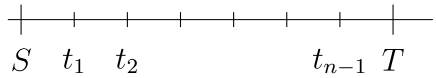
Definition 2.6 (Martingale [16]). *An n-dimensional stochastic process $\{M_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ under a probability measure $\mathbb{P}$ if:*

(i) M_t *is* $\mathcal{F}_t$-measurable for all t

(ii) $\mathrm{E}^{\mathbb{P}}\Big[|M_t|\Big] < \infty$ for all t

(iii) $\mathrm{E}^{\mathbb{P}}\Big[M_t|\mathcal{F}_s\Big] = M_s, \qquad \textit{for all } t \geq s$

The observation that Itô integrals are martingales is important in this analysis[6]. To understand why we look closer at $\int_0^T Z_u dB_u$, for some $Z_u \in \mathcal{V}[0,T]$. Assume $S \leq T$.[7]
$\mathrm{E}^{\mathbb{P}}\Big[\int_0^T Z_u dB(u)|\mathcal{F}_S\Big] = \mathrm{E}^{\mathbb{P}}\Big[\int_0^S Z_u dB(u) + \int_S^T Z_u dB(u)|\mathcal{F}_S\Big] = \mathrm{E}^{\mathbb{P}}\Big[\int_0^S Z_u dB(u)|\mathcal{F}_S\Big] +$
$\mathrm{E}^{\mathbb{P}}\Big[\int_S^T Z_u dB(u)|\mathcal{F}_S\Big] \overset{\mathcal{F}_S-measurable}{=} \int_0^S Z_u dB(u) + \mathrm{E}^{\mathbb{P}}\Big[\int_S^T Z_u dB(u)|\mathcal{F}_S\Big]$
We see that for $\int_0^T Z_u dB(u)$ to be a $\mathcal{F}_t$ martingale under $\mathbb{P}$, the expectation of the last term above must be zero. Is this the case?
For any function $f \in \mathcal{V}[0,T]$, there exist some elementary processes converging to f in $L^2(\lambda \times \mathbb{P})$-sense[8]. We slice the time interval $[S,T]$ into pieces and let $t_0 = S$, $t_n = T$.



We then take the conditional expectation of some elementary process, $\phi_u = \sum_{j=0}^{n-1} e_{t_j}(u)\chi_{(t_j,t_{j+1}]}$, with this time partition:

$$\mathrm{E}^{\mathbb{P}}\Big[\int_S^T \phi_u dB(u)|\mathcal{F}_S\Big] = \mathrm{E}^{\mathbb{P}}\Big[\sum_{j=0}^{n-1} e_{t_j}(u)[B_{t_{j+1}} - B_{t_j})|\mathcal{F}_S\Big] =$$
$$\sum_{j=0}^{n-1} \mathrm{E}^{\mathbb{P}}\Big[e_{t_j}[B_{t_{j+1}} - B_{t_j})|\mathcal{F}_S\Big] \overset{''Tower''\ \mathcal{F}_S \subset \mathcal{F}_{t_j}}{=}$$
$$\sum_{j=0}^{n-1} \mathrm{E}^{\mathbb{P}}\Big[\mathrm{E}^{\mathbb{P}}\Big[e_{t_j}[B_{t_{j+1}} - B_{t_j})|\mathcal{F}_{t_j}\Big]|\mathcal{F}_S\Big] \overset{e_{t_j}\ is\ \mathcal{F}_{t_j}-measurable}{=}$$
$$\sum_{j=0}^{n-1} \mathrm{E}^{\mathbb{P}}\Big[e_{t_j}\mathrm{E}^{\mathbb{P}}\Big[B_{t_{j+1}} - B_{t_j})|\mathcal{F}_{t_j}\Big]|\mathcal{F}_S\Big] \overset{\Delta B_{t_j}\ independent\ of\ \mathcal{F}_{t_j}}{=}$$
$$\sum_{j=0}^{n-1} \mathrm{E}^{\mathbb{P}}\Big[e_{t_j}\mathrm{E}^{\mathbb{P}}[B_{t_{j+1}} - B_{t_j})]|\mathcal{F}_S\Big] = 0$$

[6]This holds true for *any* $f \in \mathcal{V}$. We can *extend* the class of Itô-integrable functions and we denote this class by $\mathcal{W}$. When the integrand $f \in \mathcal{W}$ we have a *local martingale*. We will only refer to local martingales occasionally, for reference see [16], [21].
[7]Most of the argument is due to [22].
[8]Here λ denotes the Lebesgue measure on $[0,T]$.

Since the conditional expectation "into the future" of the Itô-integral of an elementary process is zero, this is also the case for any $f \in \mathcal{V}$ since any $f \in \mathcal{V}[0,T]$ can be written as the limit of an elementary processes in $L^2(\lambda \times \mathbb{P})$. The calculation above gives us some understanding of integration with respect to Brownian motion and shows why the important result that Itô integrals are martingales is true. Is the converse true? If we have a martingale, can it be related to an Itô integral?

Theorem 2.2 (The Martingale Representation Theorem [16]). *Suppose M_t is an $\mathcal{F}_t$-martingale (with respect to $\mathbb{P}$) and that $M_t \in L^2(\mathbb{P})$ for all $t \geq 0$. Then there exists a unique stochastic process $g(s,\omega)$ such that $g \in \mathcal{V}(0,t)$ for all $t \geq 0$ and*

$$M_t(\omega) = \mathrm{E}[M_0] + \textstyle\int_0^t g(s,\omega) dB_s$$

If we have a martingale, we know that it can be written as a pure Itô integral[9]. Our approach to modeling financial markets is by martingale modeling and now we have established a connection between no-arbitrage and Itô integrals.

Later, we will model the economy by *Itô diffusions.*

Definition 2.7 (Diffusion Process [16]). A (time-homogeneous) Itô diffusion is a stochastic process $X_t(\omega) = X(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$ satisfying a SDE of the form:

$$dX_t = b(X_t)dt + a(X_t)dB_t \tag{2.3}$$

where B_t is m-dimensional and a, b are bounded by linear growth and Lipschitz.

Since we will use diffusions as a mathematical tool for modeling the economy, we should understand how close the connection between Itô processes and Itô diffusions is.

Proposition 2.1. *We can always rewrite an Itô process into an Itô diffusion by increasing dimensionality from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$*

Proof. From Definition 2.2, an Itô process is on the form (2.1). We see that the coefficients for the drift and Brownian part can be time-dependent. On the other hand, from Definition 2.7 and eq. (2.3) we know that a *time-homogeneous* Itô diffusion does not have time-dependent coefficients. Define the Itô diffusion $Y_t = Y_t^{(s,x)}$ in $\mathbb{R}^{n+1}$ by

$$Y_t := \begin{pmatrix} s+t \\ X_t^x \end{pmatrix} \Rightarrow dY = \begin{pmatrix} 1 \\ b \end{pmatrix} dt + \begin{pmatrix} 0 \\ a \end{pmatrix} dB_t \Rightarrow dY = \tilde{b}dt + \tilde{a}dB_t, \text{ where}$$

$$\tilde{b} = \begin{pmatrix} 1 \\ b \end{pmatrix} \text{ and } \tilde{a} = \begin{pmatrix} 0 \\ a \end{pmatrix}.$$

[9] This does not just hold when the driving noise is Brownian motion but also holds for Lévy processes in general (with some restrictions on the jumps) where the representation is with respect to the Lévy-measure. (See [23]).

We recognize that Y_t is an Itô diffusion starting at $y = (s, x)$. $\square$

Later we will see how crucial it is to be able to relate a *deterministic* second-order PDE to the pricing problem. In this context, the *generator* of a diffusion will be crucial.

Definition 2.8 (Generator of an Itô diffusion [16])**.** *Let* $\{X_t\}_{t\geq 0}$ *be a (time-homogeneous) Itô diffusion in* $\mathbb{R}^n$. *The (infinitesimal) generator* A *of* X_t *is defined by*

$$Af(x) := \lim_{t\downarrow 0} \tfrac{\mathrm{E}^x[f(X_t)]-f(x)}{t};\ x \in \mathbb{R}^n$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ *such that the limit exists at* x *is denoted by* $\mathcal{D}_{\mathcal{A}}(x)$, *while* $\mathcal{D}_{\mathcal{A}}$ *denotes the set of functions for which the limit exists for all* $x \in \mathbb{R}^n$.

It will be of importance to us relating infinitesimal operators with differential operators. We will do this rather informal. The interested reader may be referred to [18, p.303-305] for a proper analysis of the domains of definition.

2.2 The Numeraire Portfolio

A numeraire is a self-financing portfolio whose value is always positive. In Brigo and Mercurio[24] we learn that Géman et al.(1995) showed that no-arbitrage between assets with spot price $S_i(t), i = 0, 1, 2, ..., n$ implies that for each numeraire with spot price $N_j, j = 0, 1, 2, ..., m$ there exist a probability measure $\mathbb{Q}^j$ equivalent to $\mathbb{P}$ such that $\frac{S_i(t)}{N_j(t)}$ is a $\mathbb{Q}^j$ martingale. Intuitively, a numeraire is a reference asset that is chosen as to normalize all other asset prices with respect to it. In a paper from 1990, John B. Long Jr. concludes:

"an asset list offers no profit opportunities if and only if a numeraire portfolio can be formed from this list. A numeraire portfolio is defined to be a self-financing portfolio such that, if current and future asset prices and dividends are denominated in units of the numeraire the expected rate of return of every asset on the list is always equal to zero."[25, p.30]

Long's observation is telling us that if asset prices are deflated by a specific portfolio, then deflated asset prices evolves as martingales under the real-world probability measure. We call this portfolio *Long's numeraire.*

2.3 Mathematical Model of Financial Markets

To explain the significance of this observation in a model of a financial market, we need some definitions of important financial concepts. First, we define what we mean by a *financial market*:

Definition 2.9 (Financial Market[16])**.** A financial market $\mathcal{M}$ is an $\mathcal{F}_t^{(m)}$-adapted $(n + 1)$-dimensional Itô process $S(t) = \Big(S_0(t), S_1(t), ..., S_n(t)\Big); 0 \leq t \leq T$.

We will assume that the market $\mathcal{M}$ is on the following form:

Assumption 1(A1) : $S_0(t)$ is the risk-free security, also called *The Money Market Account* (MMA). The MMA grows at a stochastic (short) interest rate:

$$dS_0 = r(t,\omega)S_0(t)dt; \quad S_0(0) = 1$$

Assumption 2(A2) : $S_i(t)$ is the value of the i'th risky security at time t for $i \in \{1, .., n\}$. The spot price of $S_i(t)$ evolve as a continuous real-valued semi martingale[10] over a finite time interval $[0, T]$ and the security pays no dividends.

If the market $\mathcal{M}$ is inefficient, it may lead to near-arbitrage opportunities. We will assume that our market is well-functioning and thus not offer arbitrage opportunities.

Assumption 3(A3) *There is no-arbitrage between the MMA and the n risky securities.*

A portfolio in the market $\{S(t)\}_{t\in[0,T]}$ is a $(n+1)$-dimensional, (t,ω)-measurable and $\mathcal{F}_t^{(m)}$-adapted stochastic process:

$$\theta(t,\omega) = \Big(\theta_0(t), \theta_1(t,\omega), ..., \theta_n(t,\omega)\Big).$$

Definition 2.10 (Value process [16])**.** *The value at time t of the portfolio θ is defined by:*

$$V(t,\omega) = V^{\theta}(t,\omega) := \theta(t,\omega) \cdot S(t,\omega) = \sum_{i=0}^{n} \theta_i(t) S_i(t) \tag{2.4}$$

We see that $V(t)$ is nothing more than the number of each share at time t multiplied by the corresponding value. $V(t)$ is the money value of our holdings at time t.

Definition 2.11 (Self-financing[16])**.** *The portfolio $\theta(t)$ is called self-financing if:*

$$dV(t) = \theta(t) \cdot dS(t) \text{ [11]}$$

When a portfolio possesses self-financing dynamic, we exclude that the portfolio-process is an Itô-process. For a self-financing portfolio, any change in the value process V^{θ} is due only to changes in values of the stocks and bank accounts, and not from injections or withdrawals of capital.

To explain what it means that a market does not have any arbitrage opportunities, we need to explain the concept of an *admissible*, or *tame* portfolio.

[10]A real valued process X_t defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is called a *semi-martingale* if it can be decomposed as
$X_t = M_t + A_t$
where M_t is a local martingale and A_t is c*à*dl*à*g, that is right continuous with left limits, adapted process of locally bounded variation.[10]

[11]and the appropriate integration-conditions are satisfied. These can be found in [16],[21].

Definition 2.12 (Admissible portfolio [16]). *A portfolio which is self-financing is called admissible if the corresponding value process $V^\theta(t)$ is (t, ω) a.s. lower bounded, i.e. there is some finite number K such that:*

$$V(t, \omega) \geq -K \text{ for almost all (a.a.}^{12}\text{) } (t, \omega) \in [0, T] \times \Omega$$

This restriction makes intuitively sense. In real life there must be a limit for how much debt the creditors can tolerate. If this condition is not satisfied, we obtain a result comparable to the famous "Saint Petersburg Paradox." Portfolios generating some arbitrary positive amount with probability one can then be formed[13].

We now have the definitions in place to formalize the concept of an arbitrage opportunity:

Definition 2.13 (Arbitrage [16]). *An admissible portfolio $\theta(t)$ is called an arbitrage in the market $\mathcal{M}$ if the corresponding value process $V^\theta(t)$ satisfies:*

$$\begin{gathered} V^\theta(0) = 0 \\ V^\theta(T) \geq 0 \text{ a.s. } \textit{and} \\ \mathbb{P}\Big[V^\theta(T) > 0\Big] > 0 \end{gathered}$$

If the market has arbitrage opportunities, we can generate a profit without any risk of losing money. Existence of an arbitrage is a sign of lack of equilibrium in the market. No real market equilibrium can exist in the long run if there are arbitrage opportunities there.

From the fundamental theorem of asset pricing, the market does not allow for any arbitrage opportunities if and only if there exists at least one equivalent martingale measure $\mathbb{Q}$[14]. More formally, in an arbitrage-free market we can find a $\mathbb{P}$ martingale M which can be used to create a new probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ through:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = M_T \tag{2.5}$$

Carr and Yu's model can be used to forecast the underlying of any derivative security, even if it is not traded. In a such market, any claim can not be replicated and thus the market is not complete. As we will see, being able to associate a PDE to assets is crucial. Black and Scholes derived their PDE under the assumption of a complete and arbitrage-free

[12]In probability theory, we say that *almost all* the elements of a set A have a certain property if the subset of A for which the property fails has measure zero.

[13]Karatzas and Shreve give a fascinating example in a market driven by a Brownian motion.[20, p.8-9]

[14]If there exists only one equivalent martingale measure, then the market is in addition complete.[10, Theorem 4.15] We say that a financial market is complete if any claim $F(\omega) \in L^2(\mathbb{Q})$ can be hedged. Hedging a claim means that we by an investment in the MMA and in the underlying assets are able to replicate the value of the claim.

financial market. The possibility to replicate *any square-integrable* claim by trading in the underlying assets and the MMA is essential in the derivation of the now famous Black and Scholes PDE. Since we do not necessary have the luxury of analysing a complete financial market, we need a different approach to associate a PDE to financial instruments.

By the no-arbitrage pricing paradigm, the price of any contingent claim ξ with maturity T is given by the conditional expectation:

$$\mathcal{P}_t = \mathrm{E}^{\mathbb{Q}}\Big[\exp\{-\int_t^T r_s ds\}\xi|\mathcal{F}_t\Big] \tag{2.6}$$

If we further suppose that the claim ξ is on the form $\xi = f(r_T)$ and also assuming the process $r(t)$ to be Markovian[15] (2.6) becomes:

$$\mathcal{P}_t = \mathrm{E}^{\mathbb{Q}}\Big[\exp\{-\textstyle\int_t^T r_s ds\}f(r_T)|r_s, 0 \leq s \leq t\Big] = \mathrm{E}^{\mathbb{Q}}\Big[\exp\{-\textstyle\int_t^T r_s ds\}f(r_T)|r_t\Big]$$

We define:

$$F(t,r) = \mathrm{E}^{\mathbb{Q}}\Big[\exp\{-\int_t^T r_s ds\}f(r_T)|r_t = r\Big] \tag{2.7}$$

"The analysis of the classical Black-Scholes-Merton derivative pricing theory has taught us that, since prices are given by expectations with respect to an equivalent martingale measure, they are solutions of a partial differential equation whenever the underlying dynamics are given by a Markov process under the risk-neutral martingale measure."[26, p.55]

We wish to link a PDE to (2.7). In Black and Scholes derivation of the pricing PDE we rely on completeness of the financial market. When the financial market of interest is not complete, we can not replicate any square-integrable claim and hence the derivation of the Black and Scholes PDE is not valid. But if we apply the Feynman-Kac formula[16] to the Markov process $\{r_t; t \geq 0\}$ whose dynamics under $\mathbb{Q}$ are given by the following SDE,

$$dr_t = b(t, r_t)dt + a(t, r_t)dB_t \tag{2.8}$$

we get:

Proposition 2.2 (Pricing PDE[26]). *The no-arbitrage price at time t of any contingent claim ξ of the form $\xi = f(r_T)$ with maturity $T > t$ is of the form $F(t, r_t)$ where F is a solution of the parabolic equation:*

$$\frac{\partial F}{\partial t}(t,r) + b(t,r)\frac{\partial F}{\partial r}(t,r) + \frac{1}{2}a^2(t,r)\frac{\partial^2 F}{\partial^2 r}(t,r) - rF(t,r) = 0 \tag{2.9}$$

with the terminal condition $F(T,r) \equiv f(r)$

[15] When a process X_t possesses the *Markov property* the future behaviour of the process given what has happened up to time t is the same as the behaviour obtained when started at X_t.[16]

(2.9) is often referred to as the *extended generator of the Itô diffusion in* (2.8),

$$\mathcal{G}^e_{x,t} := \hat{A} - r(x,t) \tag{2.10}$$

where $\hat{A} = \frac{\partial}{\partial t} + A$ is the generator associated to the diffusion after the transformation introduced in Proposition 2.1. A is the infinitesimal generator introduced in Definition 2.8.

From assumption **A1** $-$ **A3** we have the existence of an equivalent martingale measure $\mathbb{Q}$, under which the security prices discounted by the MMA evolve as martingales.

$$\mathrm{E}^{\mathbb{Q}}\Big[\frac{S_i(T)}{S_0(T)}|\mathcal{F}_t\Big] = \frac{S_i(t)}{S_0(t)}, t \in [0,T], i \in \{0,1,...,n\} \tag{2.11}$$

Since $d\mathbb{Q} = M_T d\mathbb{P}$, using Bayes rule for conditional expectations and the martingale property of the Radon-Nikodym derivative M_T[16] under $\mathbb{P}$, the expression above can be written as:

$$\mathrm{E}^{\mathbb{Q}}\Big[\frac{S_i(T)}{S_0(T)}|\mathcal{F}_t\Big] = \frac{\mathrm{E}^{\mathbb{P}}\Big[\frac{S_i(T)}{S_0(T)}M_T|\mathcal{F}_t\Big]}{\mathrm{E}^{\mathbb{P}}\Big[M_T|\mathcal{F}_t\Big]} = \frac{\mathrm{E}^{\mathbb{P}}\Big[\frac{S_i(T)}{S_0(T)}M_T|\mathcal{F}_t\Big]}{M_t} \Rightarrow$$
$$\mathrm{E}^{\mathbb{P}}\Big[\frac{S_i(T)}{S_0(T)}M_T|\mathcal{F}_t\Big] = M_t \mathrm{E}^{\mathbb{Q}}\Big[\frac{S_i(T)}{S_0(T)}|\mathcal{F}_t\Big] \Rightarrow$$

$$\mathrm{E}^{\mathbb{P}}\Big[\frac{S_i(T)}{S_0(T)}\frac{M(T)}{M(t)}|\mathcal{F}_t\Big] = \frac{S_i(t)}{S_0(t)}, t \in [0,T], i \in \{0,1,...,n\} \tag{2.12}$$

We next assume that there is a strong connection between the assets in our economy. More precise, we suppose that there exists a single *driver* of uncertainty, and that all financial instruments depend on this underlying process.

Assumption 4(A4) *There exists a one-dimensional time-homogeneous bounded regular*[17] *diffusion process* X_t *under the probability measure* $\mathbb{Q}$ *such that for* $i \in \{0,1,2,...,n\}, S_i(t) = S_i(t, X_t)$ *for some function* $S_i(t,x), S_i : \big([l,u] \times [0,T]\big) \to \mathbb{R}$.

Since the driver X_t evolves as a (time-homogeneous) diffusion under $\mathbb{Q}$, there exists a $\mathbb{Q}$ standard Brownian motion $B^{\mathbb{Q}}_t$, a drift function $b(x), x \in [l,u]$ and a variance rate function $a^2(x) > 0, x \in [l,u]$ such that X_t solves the following SDE:

$$dX_t = b(X_t)dt + a(X_t)dB^{\mathbb{Q}}_t \tag{2.13}$$

This assumption could potentially impose a big restriction. When modeling forward rates in an infinite dimensional stochastic analysis perspective, we ask the question of when there

[16] Here we must assume M_t to be a martingale. This is guaranteed if the *Novikov condition* holds. If M_t is a *local martingale*, care must be taken. If a *local martingale* is bounded form below we have a supermartingale; i.e. $\mathrm{E}[M^{loc}_T|\mathcal{F}_t] \leq M_t$. If a *local martingale* is bounded form above we have a submartingale; $\mathrm{E}[M^{loc}_T|\mathcal{F}_t] \geq M_t$.

[17] A process is said to be *regular* if starting from any point in the interior of the domain, any other point can be reached with positive probability.[18]

exists a finite-dimensional realization driving the forward rate curve. In a Heath-Jarrow-Morton(HJM) framework, the finite dimensional realization problem forced us to analyse forward rates with a specific shape and dynamic.[27]
We have just come up with an example of an economic phenomenon where restricting to an underlying process with finite dimensions must be handled with care and is not feasible for any HJM model. Therefore, assuming all assets under consideration are driven by one-dimensional uncertainty might be very unrealistic. Carr and Yu also share this concern.[13] They mention that for realism, one usually assumes two or three Markovian state variables driving some curve or surface, rather than assuming a one-dimensional driving process.

The assumption of assets being driven by a diffusion rather than a more general process is luckily not a big restriction. With reference to Proposition 2.1, we can always rewrite an Itô process into an Itô diffusion by increasing the dimensionality. But since we will restrict ourselves to model the driving process as a one-dimensional diffusion we can not perform this transformation and hence we must start out with a process where the coefficients do not depend on time.

We have mentioned the properties of a *numeraire portfolio* previously. Long's numeraire is the numeraire making the real-world measure $\mathbb{P}$ into a martingale measure. More formally:

Definition 2.14 (Long's numeraire portfolio[13])**.** Long's numeraire portfolio, L_t, is a strictly positive self-financing portfolio such that $\frac{S_i(t)}{L_t}$ is a martingale under $\mathbb{P}$ for all $i \in \{0, 1, 2, ..., n\}$.

We want to link the volatility of Long's numeraire to the market price of risk. If we look closely at (2.12) we are tempted to define $L_t := \frac{S_0(t)}{M_t}$. If we do so and multiply (2.12) with M_t[18] and use the definition of L_t we get:

$$\mathrm{E}^{\mathbb{P}}\Big[\frac{S_i(T)}{L_T}|\mathcal{F}_t\Big] = \frac{S_i(t)}{L_t}, i \in \{0, 1, 2, , , , n\}, t \in [0, T] \tag{2.14}$$

Thus, by defining L_t in this way we have found a promising candidate for a numeraire portfolio satisfying Long's criteria in Definition 2.14. L_t is clearly positive, from Equation (2.14), $\frac{S_i(t)}{L_t}$ is a martingale under $\mathbb{P}$ so the only thing that remains to check is if this portfolio is self-financing.

Proposition 2.3 ([13])**.** *$L_t := \frac{S_0(t)}{M_t}$ is self-financing.*

Proof. Let Y_t be an Itô process on the form: $dY_t = \zeta_t dt + dB_t^{\mathbb{P}}$, where $B_t^{\mathbb{P}}$ is a Brownian motion under $\mathbb{P}$. Set $M_t = \exp\{-\int_0^t \zeta_s dB_s - \frac{1}{2}\int_0^t \zeta_s^2 ds\}$, $0 \leq t \leq T$. If we define the measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $dQ = M_T dP$ we get from Girsanov's theorem that Y_t is a Brownian motion under $\mathbb{Q}$.

[18] M_t is $\mathcal{F}_t$-measurable and can therefore be moved inside the expectation and cancel out the $\frac{1}{M_t}$ term.

When we apply Itô's transformation formula to M_t we get:

$$dM_t = M_t\Big(-\zeta_t dB_t^{\mathbb{P}} - \tfrac{1}{2}\zeta_t^2 dt\Big) + \tfrac{1}{2}M_t\Big(-\zeta_t dB_t^{\mathbb{P}} - \tfrac{1}{2}\zeta_t^2 dt\Big)^2 = -M_t\zeta t dB_t^{\mathbb{P}}.$$

With reference to the Martingale Representation Theorem (Theorem 2.2) and the connection we established earlier between martingales and Itô integrals[19], we see that M_t is a (local) martingale under $\mathbb{P}$[20].

The reciprocal $\frac{1}{M_t}$ is a positive (local) martingale under $\mathbb{Q}$. To convince ourselves of this, we use Itô's formula on $\frac{1}{M_t} \Rightarrow$

$$d(\tfrac{1}{M_t}) = -\tfrac{1}{M_t^2}dM_t + \tfrac{1}{2}(-1)(-2)\tfrac{1}{M_t^3}(dM_t)^2 = -\tfrac{1}{M_t^2}(-\zeta_t M_t dB_t^{\mathbb{P}}) + \tfrac{1}{M_t^3}(-\zeta_t M_t dB_t^{\mathbb{P}})^2 =$$
$$\tfrac{1}{M_t}\zeta_t dB_t^{\mathbb{P}} + \tfrac{1}{M_t}\zeta_t^2 dt = \tfrac{1}{M_t}\zeta_t(dB_t^{\mathbb{P}} + \zeta_t dt) = \tfrac{1}{M_t}\zeta_t dY_t = \tfrac{1}{M_t}\zeta_t dB_t^{\mathbb{Q}}.$$

We then apply Itô's formula to the function $g(x,y) = xy$, with $x = \frac{1}{M_t}$ and $y = S_0(t)$. By doing so we discover that L_t has the following $\mathbb{Q}$ dynamics:

$$dL_t = \tfrac{S_0(t)}{M_t}\zeta_t dB_t^{\mathbb{Q}} + \tfrac{1}{M_t}S_0(t)r_t dt + d(\tfrac{1}{M_t})dS_0(t) =$$
$$L_t(\zeta_t dB_t^{\mathbb{Q}} + r_t dt) + L_t\zeta_t r_t dB_t^{\mathbb{Q}} dt = L_t(\zeta_t dB_t^{\mathbb{Q}} + r_t dt).$$

When showing L_t is self-financing, the observation that the self-financing property is invariant under normalization will be useful. That is:
$dV^\theta(t) = \theta(t)dS_t \iff d\bar{V}^\theta(t) = \theta(t)d\bar{S}(t)$, where $\bar{V}_t^\theta := \theta(t)\cdot\bar{S}(t)$[21].
When we consider Long's numeraire in the normalized market, $\tilde{L}_t := \frac{L_t}{S_0(t)}$,
we discover that:

$$d\tilde{L}_t = \tfrac{1}{S_0(t)}dL_t - \tfrac{L_t}{S_0^2(t)}dS_0(t) = \tfrac{L_t}{S_0(t)}(\zeta_t dB_t^{\mathbb{Q}} + r_t dt) - \tfrac{L_t}{S_0(t)}r_t dt = \tilde{L}_t\zeta_t dB_t^{\mathbb{Q}}$$

The only change in $\tilde{L}_t$ is due to changes in the driving process and not from money being brought in or taken out of the system. Hence $\tilde{L}(t)$ is self-financing and therefore also L_t. $\square$

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[19] We have an analogous result to the Martingale Representation Theorem for local martingales, i.e. the Local Martingale Representation Theorem. This theorem states that any local martingale can be written as an Itô integral, where the integrand is in the class of extended Itô integrable functions $\mathcal{W}$.[28]

[20] depending on whether $-\zeta_t M_t dB_t \in L^2(\lambda \times \mathbb{P})$ or not.

[21] Here $\bar{S}(t) := \frac{S(t)}{S_0(t)}$; the assets discounted by the value of the risk-free asset. Øksendal proves:

$$dV^\theta(t) = \theta(t)dS(t) \Rightarrow d\bar{V}^\theta(t) = \theta(t)d\bar{S}(t).$$

We therefore need to show the opposite implication.
Suppose $d\bar{V}^\theta(t) = \theta(t)d\bar{S}(t)$. A straightforward calculation shows that $d\bar{S}(t) = \frac{1}{S_0(t)}[dS(t) - r(t)S(t)dt]$. We then have:

$$dV^\theta(t) = d\big(\tfrac{V^\theta(t)}{S_0(t)}S_0(t)\big) = d\big(\bar{V}^\theta(t)S_0(t)\big) = d\bar{V}^\theta(t)S_0(t) + \bar{V}^\theta(t)dS_0(t) =$$
$$\theta(t)\tfrac{1}{S_0(t)}[dS(t) - r(t)S(t)dt]S_0(t) + \bar{V}^\theta(t)S_0(t)r(t)dt = \theta(t)dS(t) - \theta(t)r(t)S(t)dt + \theta(t)\tfrac{S(t)}{S_0(t)}S_0(t)r(t)dt =$$
$$\theta(t)dS(t)$$

L_t is thus the numeraire portfolio introduced by Long. Further, we have: *"The existence of the numeraire portfolio is just a simple consequence of the existence of $\mathbb{Q}$. Whenever the numeraire portfolio exists, so does the risk-neutral measure."*[13, p.50]
We want to use the connectedness between $\mathbb{P}, \mathbb{Q}$ and L to find some relation between the market price of risk and the numeraire portfolio. If such a relation can be established, finding the shape and dynamics of Long's numeraire portfolio will help us determining the risk premium. If we find the component determining the Radon-Nikodym derivative linking $\mathbb{Q}$ and $\mathbb{P}$, we have a way of obtaining the natural probabilities from the risk-neutral density. Thus we continue our quest with finding the dynamic of L_t.

Proposition 2.4 (Dynamics of Long's numeraire portfolio[13])**.** *L_t has the following dynamics under $\mathbb{P}$*[22]:

$$\tfrac{dL_t}{L_t} = (r_t + \sigma_t^2)dt + \sigma_t dB_t^{\mathbb{P}}$$

Proof. (2.14) holds for all assets. Therefore $\frac{S_0(t)}{L_t}$ is a martingale under $\mathbb{P}$. We can now exploit the insight we gained earlier. Since $\frac{S_0(t)}{L_t}$ is a martingale we know that this expression must be written as a pure Itô integral for some adapted process. From **A1**, we know that $S_0(t)$ does not have any diffusion term. Therefore the diffusion term generating the martingale dynamic must come from L_t. We have already found that the diffusion term of L_t is on the form:

$$dL_t = L_t \zeta_t dB_t^{\mathbb{Q}}.$$

This is under $\mathbb{Q}$. Since volatility does not change under a Girsanov transformation, this must be the shape of the diffusion term also under $\mathbb{P}$. To clarify that this expresses the volatility of the numeraire process, we write the lognormal volatility of L_t as σ_t. That is $\frac{dL_t}{L_t} = \sigma_t dB_t^{\mathbb{P}}$. Since the diffusion term must come from L_t the following must be true:

$$d(\tfrac{S_0(t)}{L_t}) = -\tfrac{S_0(t)}{L_t^2} L_t \sigma_t dB_t^{\mathbb{P}} = -\tfrac{S_0(t)}{L_t} \sigma_t dB_t^{\mathbb{P}} \Rightarrow$$

$$\frac{d(\frac{S_0(t)}{L_t})}{\frac{S_0(t)}{L_t}} = -\sigma_t dB_t^{\mathbb{P}} \tag{2.15}$$

Due to the martingale property we know that all the dt-terms in the expression above cancel out and hence we do not spend time showing that they actually do.
Now, let Z be an Itô process. Then:

$$d(\tfrac{1}{Z}) = -\tfrac{1}{Z^2} + \tfrac{1}{Z^3}(dZ)^2$$

Put Z equal to $\frac{S_0(t)}{L_t}$. Then:

$$d\Big(\tfrac{L_t}{S_0(t)}\Big) = -\tfrac{L_t^2}{S_0(t)^2} d\Big(\tfrac{S_0(t)}{L_t}\Big) + \tfrac{L_t^3}{S_0(t)^3}\Big(d(\tfrac{S_0(t)}{L_t})\Big)^2$$

[22] Argument inspired by [29].

When dividing both sides by $\frac{L_t}{S_0(t)}$ and substituting (2.15) in the expression above we get:

$$\frac{d(\frac{L_t}{S_0(t)})}{\frac{L_t}{S_0(t)}} = -(-\sigma_t dB_t^{\mathbb{P}}) + (-\sigma_t dB_t^{\mathbb{P}})^2 = \sigma_t dB_t^{\mathbb{P}} + \sigma_t^2 dt.$$

If we apply Itô's formula to the expression $\frac{L_t}{S_0(t)}$ we get:

$$d(\frac{L_t}{S_0(t)}) = \frac{1}{S_0(t)} dL_t - \frac{L_t}{S_0(t)^2} dS_0(t) + \frac{1}{2}\Big(- \frac{1}{S_0(t)^2}(dS_0(t))(dL_t) + 2\frac{L_t}{S_0(t)^3}(dS_0(t))^2\Big)$$

Since $S_0(t)$ is of finite variation, only the first order terms from Itô's formula are non-zero. Substituting $dS_0(t) = S_0(t) r_t dt$ and dividing the expression above by $\frac{L_t}{S_0(t)}$ we get

$$\frac{d(\frac{L_t}{S_0(t)})}{\frac{L_t}{S_0(t)}} = \frac{dL_t}{L_t} - r_t dt \Rightarrow$$
$$\frac{dL_t}{L_t} = (r_t + \sigma_t^2)dt + \sigma_t dB_t^{\mathbb{P}}$$

□

Long's observation that the numeraire portfolio always exists in an arbitrage-free market allows us to focus on the more concrete and economically intuitive concept of a numeraire portfolio rather than on the notion of an equivalent martingale measure.

Assumption 5(A5): *We assume that the numeraire portfolio, L_t, only depends on the diffusion X_t and time t. That is:*

$$L_t := L(t, X_t)$$

We also assume that the risk-neutral drift r_t and σ_t both depends on X_t and are independent of time.

Under $\mathbb{Q}$ the drift of all assets is the risk-free rate, and since volatility does not change under a Girsanov transformation, we get the following dynamic for the numeraire portfolio under $\mathbb{Q}$:

$$\frac{dL_t}{L_t} = r(X_t)dt + \sigma(X_t)dB_t^{\mathbb{Q}} \tag{2.16}$$

Proposition 2.5 ([13]). *The market price of risk is just the instantaneous volatility of Long's numeraire portfolio:*

$$\Theta_t = \sigma_t$$

Proof. From Girsanov's Theorem we have that Brownian motions under respectively $\mathbb{Q}$ and $\mathbb{P}$ are related by the following equality:

$$dB_t^{\mathbb{Q}} = \Theta_t dt + dB_t^{\mathbb{P}}.$$

If we insert this for $dB_t^{\mathbb{P}}$ in the expression in Proposition 2.4 we get:

$\frac{dL_t}{L_t} = r(X_t)dt + \sigma^2(X_t)dt + \sigma(X_t)(dB_t^{\mathbb{Q}} - \Theta_t dt) = \Big(r(X_t) + \sigma^2(X_t) - \sigma(X_t)\Theta_t\Big)dt + \sigma(X_t)dB_t^{\mathbb{Q}}$.
Comparing this expression with the one in Equation (2.16) we see that $\Theta_t = \sigma(X_t)$. □

2.4 Derivation of Carr and Yu's recovery result

When combining **A5** with Proposition 2.4 we obtain the following dynamics of L_t under the real-world probability measure $\mathbb{P}$:

$$\frac{dL_t}{L_t} = \Big[r(X_t) + \sigma^2(X_t)\Big]dt + \sigma(X_t)dB_t^{\mathbb{P}} \tag{2.17}$$

Proposition 2.5 is pivotal for the analysis. If we can determine the numeraire portfolio's volatility process from data, we can determine the real-world dynamics for any contingent claim driven by X_t.

Proposition 2.6 (Market Price of Risk[13]). *The volatility of Long's numeraire, and hence the market price of risk, is on the following form:*

$$\sigma(x) = a(x)\tfrac{\partial}{\partial x}\ln L(t,x)$$

Proof. Starting with the shape as specified in **A5** of L_t, we find the dynamics to be:

$$dL_t = \tfrac{\partial L_t}{\partial t}dt + \tfrac{\partial L_t}{\partial x}dX_t + \tfrac{1}{2}\tfrac{\partial^2 L_t}{\partial^2 x}(dX_t)^2$$

Diving both sides by L_t this expression becomes:

$$\tfrac{dL_t}{L_t} = \tfrac{1}{L_t}\tfrac{\partial L_t}{\partial t}dt + \tfrac{1}{L_t}\tfrac{\partial L_t}{\partial x}dX_t + \tfrac{1}{L_t}\tfrac{1}{2}\tfrac{\partial^2 L_t}{\partial^2 x}(dX_t)^2$$

We then insert for the dynamics of X_t under $\mathbb{P}$. When comparing the diffusion term with Proposition 2.4, we discover:

$$\sigma(x) = \tfrac{1}{L(t,x)}\tfrac{\partial L(t,x)}{\partial x}a(x)$$

The proposition follows from the observation that $\frac{1}{L(t,x)}\frac{\partial L(t,x)}{\partial x} = \frac{\partial}{\partial x}\ln L(t,x)$ □

We will now reorganize the expression for $\sigma(x)$ in Proposition 2.6 so that we can apply regular Sturm-Liouville(SL) theory (Appendix A) to solve for the differential equation associated with the generator $\mathcal{G}^e_{x,t}$.

Proposition 2.7 ([13]). *The numeraire portfolio can be separated as:*

$$L(t,x) = p(t)\pi(x)$$

Proof. By assumption, $a(x) > 0$ and we can therefore divide $\sigma(x)$ by $a(x)$. Doing so we get:

$$\tfrac{\partial}{\partial x}\ln L(t,x) = \tfrac{\sigma(x)}{a(x)}.$$

If we integrate with respect to x this becomes:

$$\int_l^x \frac{\partial}{\partial y} \ln L(t,y)dy = \int_l^x \frac{\sigma(y)}{a(y)}dy \Rightarrow \ln L(t,x) - \ln L(t,l) = \int_l^x \frac{\sigma(y)}{a(y)}dy + C,$$

where C is the constant of integration. Further, let $f(t) = \ln L(t,l) + C$. We thus get: $\ln L(t,x) = \int_l^x \frac{\sigma(y)}{a(y)}dy + f(t)$, and by applying the exponential function to both sides and defining:

$$p(t) := \exp\{f(t)\} \text{ and } \pi(x) := \exp\{\int_l^x \frac{\sigma(y)}{a(y)}dy\}$$

the proposition follows.

□

Remark 1 (Market Price Risk[13])**.** When we combine the result from Proposition 2.7 with the expression for $\sigma(x)$ in Proposition 2.6 we get:

$$\sigma(x) = a(x)\frac{\partial}{\partial x}\ln\Big(p(t)\pi(x)\Big) = a(x)\frac{\partial}{\partial x}\Big(\ln p(t) + \ln\pi(x)\Big) = a(x)\frac{\partial}{\partial x}\ln\pi(x) \tag{2.18}$$

This will be useful when we later are going to investigate the market price of risk suggested by our model.

In the discussion related to (2.9) we learned that under no-arbitrage pricing theory, a PDE must be satisfied. We can therefore associate two *deterministic* differential equations to Long's numeraire portfolio.

Proposition 2.8 ([13])**.** *Under the no-arbitrage paradigm we can associate a differential equation to both* p *and* π *that must hold simultaneously. More specific:*

$$\frac{p'(t)}{p(t)} = \lambda \textit{ and}$$
$$b(x)\pi'(x) + \frac{a^2(x)}{2}\pi''(x) - r(x)\pi(x) = -\lambda\pi(x)$$

Proof. We apply the *extended generator* (2.10) to the expression for L_t in Proposition 2.7. From Proposition 2.2 we know that under no-arbitrage, $\mathcal{G}^e_{x,t}L(x,t) = 0$.

$\mathcal{G}^e_{x,t}L(t,x) = 0 \Rightarrow \pi(x)p'(t) + b(x)\pi'(x)p(t) + \frac{a^2(x)}{2}\pi''(x)p(t) - r(x)\pi(x)p(t) = 0$

Since L_t is non-zero, so must $p(t)\pi(x)$ be. Therefore we can divide the expression above by $\pi(x)p(t)$. Rearranging yields:

$$b(x)\frac{\pi'(x)}{\pi(x)} + \frac{a^2(x)}{2}\frac{\pi''(x)}{\pi(x)} - r(x) = -\frac{p'(t)}{p(t)}$$

The left-hand side depends only on the variable x, while the right-hand side involves only the variable t. This is consistent only if both sides are equal to a constant. Denote this constant by $-\lambda$.

Hence:

$$\frac{p'(t)}{p(t)} = \lambda \tag{2.19}$$

and

$$b(x)\pi'(x) + \frac{a^2(x)}{2}\pi''(x) - r(x)\pi(x) = -\lambda\pi(x) \tag{2.20}$$

□

If we are able to find the solutions of the deterministic differential equations in Proposition 2.8 we have the explicit expression for L_t. If we then apply the result form Remark 1, we have determined the market price of risk.

Theorem 2.3 (Recovery Theorem[13])**.** *The only valid solution of the differential equations in Proposition 2.6 is $L(t,x) = p(t)\pi(x) = e^{\rho t}\phi(x)$ where $\phi(x)$ is the first eigenfunction with corresponding eigenvalue $\lambda = \rho$.*

Proof. Equation (2.19) is easy to solve, the observation $\frac{p'(t)}{p(t)} = \frac{d}{dt}\ln p(t)$ is all that is needed. Doing so we find the solution of (2.19) to be: $p(t) = p(0)\exp\{\lambda t\}$. Assume $p(0) = 1$.

Solving Equation (2.20) is harder. With reference to Appendix A, we note that (2.20) can be written as:

$$\tfrac{\partial}{\partial x}\left(z(x)\tfrac{\partial \pi(x)}{\partial x}\right) - \tfrac{2r(x)}{a^2(x)}\pi(x)z(x) = -\tfrac{2\lambda}{a^2(x)}\pi(x)z(x)$$

where $z(x) = \exp\{\int_l^x \frac{2b(y)}{a^2(y)}dy\}$.
If we define $q(x) := \frac{2z(x)r(x)}{a^2(x)}$ and $\rho(x) := \frac{2z(x)}{a^2(x)}$ and write (2.20) as:

$$\tfrac{\partial}{\partial x}\left(z(x)\tfrac{\partial \pi(x)}{\partial x}\right) - q(x)\pi(x) = -\lambda\rho(x)\pi(x)$$

we see that this differential equation is on SL form as in (A.1). Theory suggest that the first eigenfunction $\phi(x)$ is positive and all other eigenfunctions switch sign at least once (see Appendix A, property 3.). Since the numeraire portfolio is strictly positive for *any* value the diffusion may take, we must have that $\pi(x) = \phi(x)$ with the associate first eigenvalue. Therefore $L(t,x) = e^{\rho t}\phi(x)$. □

From Appendix A we note that the boundary conditions are an important part of a regular SL problem. In order to solve eq. (2.20) and recover, we have to impose some rather awkward boundary conditions on Long's numeraire portfolio. In section 3 we are forced to investigate boundary conditions more carefully.

The result in Theorem 2.3 can be used to determine the real-world probability distribution of the diffusion X_t.

Proposition 2.9 ([13])**.** *The real-world probability measure $\mathbb{P}$ is related to the risk-neutral probability measure $\mathbb{Q}$ by the following relationship:*

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{-\int_0^T r(X_s)ds} \frac{\phi(X_T)}{\phi(X_0)} e^{\rho T}$$

Proof. From the change of numeraire theory[23] and using $L(t,x) = e^{\rho t}\phi(x)$ we get that $\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{S_0(0)}{S_0(T)} \frac{L_T}{L_0} = e^{-\int_0^T r(X_s)ds} \frac{L(T,X_T)}{L(0,X_0)} = e^{-\int_0^T r(X_s)ds} \frac{\phi(X_T)}{\phi(X_0)} e^{\rho T}$. □

Next, determine the real-world dynamics for the stocks.

Proposition 2.10. *Under the real-world probability measure $\mathbb{P}$ the risky assets have the following dynamics:*

$$\frac{dS_i(t,X_t)}{S_i(t,X_t)} = \Big[r(X_t) + \sigma(X_t)a(X_t)\frac{\partial}{\partial x}\ln S_i(t,X_t)\Big]dt + a(X_t)\frac{\partial}{\partial x}\ln S_i(t,X_t)dB_t^{\mathbb{P}}$$

Proof. First, from **A4**, $dX_t = b(X_t)dt + a(X_t)dB_t^{\mathbb{Q}}$.
Since $dB_t^{\mathbb{Q}} = \Theta_t dt + dB_t^{\mathbb{P}}$ we get that $dX_t = \Big(b(X_t) + a(X_t)\Theta_t\Big)dt + a(X_t)dB_t^{\mathbb{P}}$.
From Proposition 2.5 we have $\Theta_t = \sigma_t = \sigma(X_t)$. Using this, and due to **A4**, the risky assets have the following dynamics:

$$dS_{i,t} = \frac{\partial}{\partial t}S_i(t,X_t)dt + \frac{\partial}{\partial x}S_i(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}S_i(t,X_t)dX_t^2 = \bigg(\frac{\partial}{\partial t}S_i(t,X_t) + \frac{\partial}{\partial x}S_i(t,X_t)\Big(b(X_t) + a(X_t)\sigma(X_t)\Big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}S_i(t,X_t)a(X_t)^2\bigg)dt + \frac{\partial}{\partial x}S_i(t,X_t)a(X_t)dB_t^{\mathbb{P}} = \Big(\frac{\partial}{\partial t}S_i(t,X_t) + b(X_t)\frac{\partial}{\partial x}S_i(t,X_t) + a(X_t)^2\frac{1}{2}\frac{\partial^2}{\partial x^2}S_i(t,X_t)\Big)dt + \frac{\partial}{\partial x}S_i(t,X_t)a(X_t)\sigma(X_t)dt + \frac{\partial}{\partial x}S_i(t,X_t)a(X_t)dB_t^{\mathbb{P}}.$$

Due to no-arbitrage, we know that every risky asset solves the following linear parabolic PDE: $\mathcal{G}^e_{x,t}S_i(t,x) = 0$. We have gathered terms such that the first parenthesis is equal to $r(t,x)S_i(t,x)$ and from assumption **A5** we have $r(t,x) = r(x)$. The result follows by dividing by $S_i(t,X_t)$ on both sides and using that $\frac{1}{S_i(t,X_t)}\frac{\partial}{\partial x}S_i(t,X_t) = \frac{\partial}{\partial x}\ln S_i(t,X_t)$. □

We note that $\sigma(X_t)a(X_t)\frac{\partial}{\partial x}\ln S_i(t,X_t)$ is the drift above the interest rate $r(X_t)$ and is therefore the *instantaneous risk premium*. If we calculate $d\ln S_i(t,X_t)\ d\ln L(t,X_t)$, we arrive at the expression above. In stochastic calculus, the *quadratic covariance* between two processes, X and Y, are defined by:

$$\Big[X,Y\Big]_t(\omega) = \lim_{\Delta t_k \to 0}\sum_{t_k \le t}\Big(X_{t_k} - X_{t_{k-1}}\Big)\Big(Y_{t_k} - Y_{t_{k-1}}\Big).$$

The *instantaneous risk premium* is just the quadratic covariation on returns on $S_{i,t}$, with returns on L_t.

[23] see for example Brigo and Mercurio.[24]

2.5 Illustrating graph

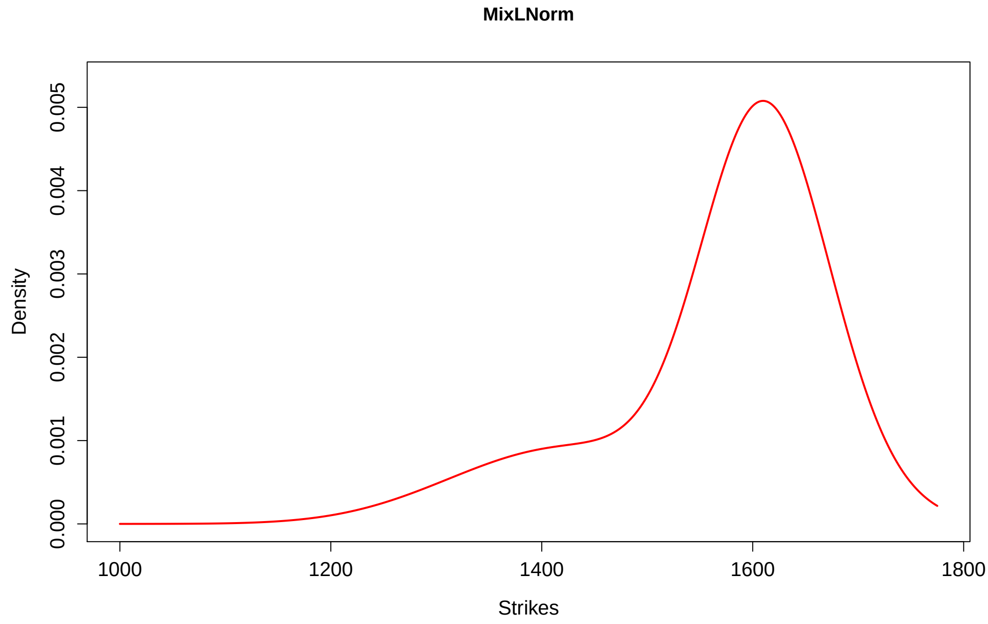

Figure 1: Risk-neutral density extracted on data from 2013-04-19 for options on S&P 500 with 62 days to expiration.

3 Unbounded diffusions

With some handwaving, Carr and Yu obtain their fascinating result by restricting the dynamics of the numeraire portfolio, assuming some set of assets is driven by a time-homogeneous *bounded* diffusion so that we can exploit properties of the solution of a regular SL problem to pin down the first eigenvalue and the first eigenfunction to determine the real-world probability density.
We wish to extend the recovery result to situations where we do not have the luxury of working with bounded diffusions. To understand more about what seems to *drive* the jaw-dropping result in Carr and Yu, we have to dig a bit deeper into SL theory.

Properties with the *spectrum* of the associated SL problem are of importance in our analysis. When the domain of the SL problem is finite[24], the SL problem is said to be *regular*. Otherwise, the problem is *singular*.

Remark 2. *"For a regular (SL) problem with two Dirchlet boundary conditions the spectrum is simple, purely discrete and strictly positive for* $r(x) \geq 0$*. In contrast, the spectrum of a*

[24]and the killing, speed and scale measure are absolutely integrable near the both end-points. We will study what this means later.

singular problem can be discrete, continuous, or mixed, and further analysis is needed to determine the nature of the spectrum in each case."[30, p.190]

Carr and Yu pull the rabbit out of their hat by using the properties of the spectrum of their separated second-order differential equation the numeraire portfolio must satisfy due to no-arbitrage. When assuming a bounded underlying diffusion, the associated SL problem is regular and then we know from Appendix A that we obtain an eigenfunction expansion. With the chosen approach, this is what allows us to recover. Since we are going to make an attempt on recovery with unbounded diffusions, understanding the SL problem seems to be at the core. The SL equation can be viewed as a linear operator mapping a function u to another function $\mathcal{L}u$,[25]and therefore we need some results from linear functional analysis to proceed .

3.1 Operator theory and spectrum

When we analyse matrices, we realize how helpful *eigenvalues* and *eigenvectors* are to characterize important properties with the matrices under scrutiny.
It is generally not straightforward to look at some matrix $\mathbf{A}$ and immediately tell what it is going to do when we multiply it with some vector $\mathbf{x}$. If we for example want to take the exponential of a matrix, using the expansion $e^{\mathbf{A}}\mathbf{x} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}\mathbf{x}$, we see that we have a formidable task of calculating all of $\mathbf{A}$'s powers and multiplying these with $\mathbf{x}$. In such a problem, the concept of an eigenbasis is of immense help. By choosing a different basis for our vector space, we can alter the appearance of the matrix $\mathbf{A}$ in that basis. Simply speaking, the i-th column of $\mathbf{A}$ tells us what the i-th basis vector multiplied with $\mathbf{A}$ would look like. If all our basis vectors are also eigenvectors, then it is not hard to see that the matrix $\mathbf{A}$ is diagonal. In our example, when $\mathbf{v}$ is an eigenvector we have that $e^{\mathbf{A}}\mathbf{v} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}\mathbf{v} = \sum_{n=0}^{\infty} \frac{\lambda^n \mathbf{v}}{n!} = e^{\lambda}\mathbf{v}$.

"Even though matrices and linear transformations between finite-dimensional vector spaces are logically distinct concepts, there is a close connection between them, and much of their theory is in essence, identical." [31, p.11]

In broad terms the spectral theorem provides conditions under which an operator or a matrix can be diagonalized.

Definition 3.1 (Linear Transformation[31])**.** Let V, W be vector spaces over the same scalar field $\mathbb{F}$. A function $T : V \to W$ is called a linear transformation if, for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in V$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

[25] $\mathcal{L}$ as in appendix A.

Definition 3.2 (Operator[31]). Let X and Y be normed linear spaces. The set of all continuous linear transformations from X to Y is denoted by $B(X, Y)$. Elements of $B(X, Y)$ are also called *bounded linear operators* or *linear operators* or sometimes just *operators.*

Given a square matrix $\mathbf{A}$, an important set of complex numbers is the set:

$$\mathcal{A} = \{\lambda \in \mathbb{C} : \mathbf{A} - \lambda\mathbf{I} \text{ is not invertible } \}$$

Here $\mathbf{I}$ is the identity matrix. In fact $\mathcal{A}$ consists of the sets of eigenvalues of $\mathbf{A}$. The spectrum of an operator is a generalization of this set.

Definition 3.3 (Spectrum[31]). Let $\mathcal{H}$ be a complex Hilbert space[26], let $\mathbb{I} \in B(\mathcal{H}, \mathcal{H})$ be the identity operator and let $T \in B(\mathcal{H}, \mathcal{H})$. The *spectrum* of T is denoted by $\sigma(T)$ and is defined to be:

$$\sigma(T) = \Big\{\lambda \in \mathbb{C} : T - \lambda\mathbb{I} \text{ is not invertible } \Big\}$$

When some conditions are satisfied, we are able to obtain a *spectral representation* for the pricing problem. Since we first of all are interested in solving a differential equation and not necessarily compute prices, we should find out how the generator $\mathcal{G}^e_{x,t}$ and the pricing operator $\mathcal{P}$ are connected. In Proposition 2.2 we learned that pricing of claims were closely related to the *extended generator.*

Under the hypothesis of the underlying process being a regular time-homogeneous diffusion, the family of pricing operators $\{\mathcal{P}_t, t \geq 0\}$ form a *semigroup* in an appropriate payoff space.[32, p.237]

Definition 3.4 (Semigroup[19]). *The operator $\mathcal{P}_t$ is said to enjoy the semigroup property if for every bounded piecewise-continuous function f, we have*:

$$\Big(\mathcal{P}_{t+s}f\Big)(x) = \Big(\mathcal{P}_t\Big(\mathcal{P}_s f\Big)\Big)(x) = \Big(\mathcal{P}_s(\mathcal{P}_t f)\Big)(x)$$
$$\Big(\mathcal{P}_0 f\Big)(x) = \Big(\mathbb{I}f\Big)(x) = f(x)$$

The semigroup is called a *contraction semigroup* if $\|\mathcal{P}_t\| \leq 1$ for $t \geq 0$. Here $\|\cdot\|$ is the operator norm defined as $\|\mathcal{P}_t\| = \sup\Big\{\|\mathcal{P}_t f\| : \|f\| \leq 1\Big\}$.

Definition 3.5 (Pricing semigroup[32]). *The pricing semigroup is:*

$$\Big(\mathcal{P}_t f\Big)(x) := \mathrm{E}^{\mathbb{Q}}\Big[\exp\{-\textstyle\int_0^t r(X_s)ds\}f(X_t)|X_0 = x\Big] = \mathrm{E}^{\mathbb{Q}}_x\Big[\exp\{-\textstyle\int_0^t r(X_s)ds\}f(X_t)\Big]$$

[26]If an *inner product* space is complete, i.e. every cauchy sequence converges, it is called a *Hilbert space.*

as in Equation (2.6), but we are a bit more careful and include the initial value of the diffusion as the argument on the left hand side. As Karlin and Taylor highlight, the theory of semigroups of operators is a powerful tool for the study of continuous time Markov processes. For us the most important property will be the link between the pricing operator and the generator. To every semigroup one can associate an operator which is called an *infinitesimal generator* of the semigroup:

Definition 3.6 ([18])**.** The infinitesimal generator of the pricing semigroup is defined as the appropriate limit of the operator sequence

$$\mathcal{D} := \lim_{h \downarrow 0} \left[\tfrac{\mathcal{P}_h - \mathbb{I}}{h} \right].$$

The meaning ascribed to the operator representation $\mathcal{P}_t = e^{\mathcal{D}t}$ requires care, and the characterization of the domain is a delicate matter.

Remark 3. *The semigroup associated to the pricing problem and the extended differential infinitesimal operator coincide.*

It will be advantageous to note that the operator $\mathcal{D}$ is the same as the extended generator $\mathcal{G}^e_{x,t}$ from Equation (2.10).
Let $\tilde{X}_t$ be the diffusion:

$$\tilde{X}_t = \begin{cases} X_t & \text{if } t < \varsigma \\ \partial & \text{if } t \geq \varsigma \end{cases}$$

where $\partial \notin \mathbb{R}^n$ is some "coffin state" and $\varsigma = \inf\Big\{ s \geq 0; \int_0^s c(X_\eta) d\eta > R \Big\}$, where R is a positive valued random variable following an exponential distribution with parameter 1 (see [18, p.313-316]). The *killing rate* $c(x)$ must be nonnegative. Now:

$$\mathrm{E}_x[f(\tilde{X}_t)] := \mathrm{E}_x[f(X_t) \cdot \chi_{[0,\varsigma)}(t)] = \mathrm{E}_x[f(X_t) \cdot e^{-\int_0^t c(X_s)ds}]$$

Specific, we have:

$$\Big(\mathcal{P}_t f\Big)(x) = \mathrm{E}_x\Big[f(X_t)e^{-\int_0^t r(X_s)ds}\Big],$$

so that r is our killing rate. *"We often use the terminologies of discounting and killing interchangeably."*[32, p26]
By the Feynman-Kac formula[16] we have that:

$$(\mathcal{D}f)(x) = \lim_{t \downarrow 0} \left(\left[\tfrac{\mathcal{P}_t - \mathbb{I}}{t} \right] f \right)(x) = \lim_{t \downarrow 0} \tfrac{\mathrm{E}_x[f(\tilde{X}_t)] - f(x)}{t} = Af(x) - r(x)f(x).$$

Here A is the infinitesimal generator from Definition 2.8. Since the extended generator is $\mathcal{G}^e_{x,t} = \frac{\partial}{\partial t} + b(x)\frac{\partial}{\partial x} + \frac{1}{2}a(x)^2\frac{\partial^2}{\partial^2 x} - r(t,x)$ we see that $\mathcal{G}^e_{x,t}$ and $\mathcal{D}$ are the same, possibly after the transformation mentioned in Proposition 2.1 with $y = (t,x)$. That is if $\tilde{X}_t$ is transformed into a diffusion by the method proposed in Proposition 2.1, we know from the argument leading to eq. (2.10) that the infinitesimal generator of $\tilde{X}_t$ will become $\hat{A} = \frac{\partial}{\partial t} + A$.

Theorem 3.1 (Spectral Theorem for Self-Adjoint Operators[32]). *From the Spectral Representation Theorem for Self-Adjoint Operators we have that there is a one-to-one correspondence between self-adjoint operators T and projection-valued measures, $\{E(B), B \in \mathcal{B}(\mathbb{R})\}$ in $\mathcal{H}$. The spectral representation theorem of a self-adjoint operator can be abbreviated as:*

$$T = \int_{\mathbb{R}} \lambda E(d\lambda)$$

The *spectrum* of T coincides with the support of its spectral measure E.[27]

Theorem 3.2 ([32]). *The operator $\mathcal{D}$ is the infinitesimal generator of a strongly continuous self-adjoint contracting semigroup $\{\mathcal{P}_t, t \geq 0\}$ in $\mathcal{H}$ if and only if $\mathcal{D}$ is a non-positive self-adjoint operator in $\mathcal{H}$. If*

$$-\mathcal{D} = \int_{[0,\infty)} \lambda E(d\lambda)$$

is the spectral representation of $-\mathcal{D}$, then for every $t \geq 0$

$$\mathcal{P}_t = e^{t\mathcal{D}} = \int_{[0,\infty)} e^{-\lambda t} E(d\lambda)$$

where $E(d\lambda)$ is the so-called projection-valued spectral measure corresponding to the negative of the infinitesimal generator $-\mathcal{D}$. *"We thus have a one-to-one correspondence between strongly continuous self-adjoint contracting semigroups in $\mathcal{H}$ and non-positive self-adjoint operators in $\mathcal{H}$, their generators."*[32, p.235] The *Spectral Representation* for the problem we are interested in now becomes:

$$\mathcal{P}_t f = \int_{[0,\infty)} e^{-\lambda t} E(d\lambda) f, \forall f \in L^2(I, m) \tag{3.1}$$

where $E(d\lambda)$ is the so-called *projection-values spectral measure* and m is some measure. When the spectrum of $-\mathcal{D} = -\mathcal{G}^e_{x,t}$ [28] is purely discrete, the spectral representation of the semigroup generated by $\mathcal{G}^e_{x,t}$ simplifies to the *eigenfunction expansion*:

$$\mathcal{P}_t f = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \varphi_n, \quad c_n = \langle f, \varphi_n \rangle\, ^{29}, \quad \forall f \in L^2(I, m) \tag{3.2}$$

where $\{\varphi_n\}_{n\in\mathbb{N}}$ are a complete orthonormal system of eigenvectors of $-\mathcal{G}^e_{x,t}$ with eigenvalues λ_n, that is, $-\mathcal{G}^e_{x,t}\varphi_n = \lambda_n \varphi_n$. The $\varphi_n's$ are also eigenvectors of $\mathcal{P}_t$ with eigenvalues $e^{-\lambda_n t}$, that is, $\mathcal{P}_t \varphi_n = e^{-\lambda_n t}\varphi_n$. They form a complete orthonormal basis in $L^2(I, m)$, and c_n are the expansion coefficients of the payoff f in this basis. If we can work out the spectral representation in a closed form for a particular model, it provides explicit closed-form representation for the pricing operator and state-price densities.

[27] The *support of a spectral measure* can be defined as the smallest closed subsets in $\mathbb{R}$ such that $E\big(Supp(E)\big) = \mathbb{I}$, where $\mathbb{I}$ is the identity operator in $\mathcal{H}$.

[28] Remark 3.

[29] with the inner product $\langle f, g \rangle = \int_I f(x)g(x)m(dx)$.

When the process is a one-dimensional diffusion on some interval $I \subset \mathbb{R}$ taken together with the speed measure $\mathfrak{m}$[30] (and with appropriate boundary conditions), the pricing semi-group is symmetric in $L^2(I, \mathfrak{m})$, and the Spectral Representation Theorem provides us with a spectral representation(see[32, p.228-229]).

3.2 Diffusion theory; killing, scale and speed measure

We now assume that some set of assets in the economy are driven by an *unbounded*, one-dimensional, time-homogeneous regular diffusion, and we therefore modify assumption **A4** in a natural manner by letting $-\infty \leq l < u \leq \infty$. We denote the state space of the diffusion as $I \subset \mathbb{R}$.
The differential equation in (2.20) can be linked with three sizes essential for a diffusion, namely *the killing rate*, *the scale density* and *the speed densitiy.* We have mentioned the role of *the killing rate* earlier.

Definition 3.7 ([18]). *The scale density is defined as:*

$$\mathfrak{s}(x) := \exp\{-\int_{x_0}^{x} \tfrac{2b(y)}{a^2(y)} dy\}$$

where x_0 is an arbitrary point in the state space. The *scale function* $\mathfrak{s}(x)$ can be used to rescale the state space (l, u) in terms of the probabilities of achieving various levels, and this use motivates the name.

Definition 3.8 ([18]). *The speed density is defined as:*

$$\mathfrak{m}(x) := \tfrac{2}{a^2(x)\mathfrak{s}(x)}$$

Similarly, the *speed density* $\mathfrak{m}(x)$ can be constructed as the speed at which the clock of the process runs when located at the state point x.
We can express *the extended generator*, (2.10), in terms of these two measures.

Proposition 3.1. *The infinitesimal generator in Definition 2.8 can be expressed in terms of the scale and speed measure, and we call this the the canonical representation of the differential infinitesimal operator. That is, we can write the infinitesimal generator as:*

$$Af(x) = \tfrac{d}{dM}\left[\tfrac{df(x)}{dS}\right], \quad dM = \mathfrak{m}(x)dx, dS = \mathfrak{s}(x)dx$$

In particular, the problem in Proposition 2.2, $\left(\mathcal{G}^e_{x,t} f\right)(x) = 0$, *translates into*

$$\tfrac{d}{dM}\left[\tfrac{df(x)}{dS}\right] = g(x), \text{ where } g(x) = r(x)f(x).$$

Proof. From Definition 3.8, we observe $\mathfrak{s}'(x) = \mathfrak{s}(x)(-\frac{2b(x)}{a^2(x)})$. We apply the *infinitesimal differential operator* to an arbitrary $f \in C^2(\mathbb{R})$ and straightforward calculations shows:

[30] See Definition 3.8.

$$\Big(Af\Big)(x) = \frac{a^2(x)}{2} f''(x) + b(x)f'(x) = \frac{a^2(x)}{2}\frac{\mathfrak{s}^2(x)}{\mathfrak{s}^2(x)} f''(x) + \frac{a^2(x)}{a^2(x)}\frac{2}{2}\frac{\mathfrak{s}^2(x)}{\mathfrak{s}^2(x)} b(x)f'(x) =$$

$$\frac{a^2(x)}{2}\frac{\mathfrak{s}^2(x)}{\mathfrak{s}^2(x)} f''(x) + \frac{a^2(x)\mathfrak{s}^2(x)}{2}\frac{f'(x)\frac{2b(x)}{a^2(x)}}{\mathfrak{s}^2(x)} = \frac{1}{\frac{2}{a^2(x)\mathfrak{s}(x)}}\left(\frac{f''(x)\mathfrak{s}(x) - f'(x)\mathfrak{s}(x)\Big(-\frac{2b(x)}{a^2(x)}\Big)}{\mathfrak{s}^2(x)}\right) = \frac{1}{\mathfrak{m}(x)}\left(\frac{f'(x)}{\mathfrak{s}(x)}\right)'.$$

The first result follows from writing $f'(x) = \frac{df(x)}{dx}$, similiary for $(\cdot)' = \frac{d}{dx}(\cdot)$, and using the fact that $\mathfrak{s}(x) = \frac{dS}{dx}$ and $\mathfrak{m}(x) = \frac{dM}{dx}$. The last observation in Proposition 3.1 follows from eq. (2.10).

□

When we have boundary conditions for f, the solution of *the canonical representation of the differential infinitesimal operator* follows directly from two successive integrations. For this reason, knowing $M(l), M(u), S(l)$ and $S(u)$ is essential.

Absorbed Brownian motion is a regular diffusion defined on the state space $I = [0, \infty)$. Starting from a point $X(0) = x_0$ in the interior of the interval, that is $x_0 > 0$, the process acts like Brownian motion until the level zero is first reached. Therefore, the parameters of the diffusion are the same as for Brownian motion when we are in the interior of the state space. For this reason, we realize that these parameters do not uniquely define a diffusion, and to fully characterize a diffusion we need to specify the behaviour at any endpoint of I. It turns out that the *scale*, *speed* and *killing* measure are important for determining boundary behaviour, and this is the reason why we have brought them into our analysis.

3.3 Feller's boundary classification

Study of diffusions arose as an attempt to understand physical phenomena. As an example, the mathematical model of Brownian motion tried to explain the ceaseless irregular motions of tiny particles suspended in a fluid. When picturing a moving particle, we can easily imagining this particle hitting some barrier and bouncing back or hitting something trapping it. With this in mind it is not surprising that there is an extensive theory of boundary classification of diffusion processes. To be able to solve (2.20) for an unbounded diffusion we need to know more about the behaviour at the boundaries.

Modern classification of boundary behaviour is based on four functionals[31] and whether or not they are finite or infinite. Karlin and Taylor provide a brilliant but lengthy exposition of why this is true, in this text we will not have the luxury of spending enough time with the concepts to understand why this is so, but since these functionals are crucial for the properties of the solution of (2.20) with unbounded diffusions, we will spend some time trying to gain insight and intuition behind *why* these functionals are so important and what they are trying to measure.

[31] A functional is a function of a function.

In stochastic modeling, the *first hitting time* is an important concept. For any point z in I, let $\mathbf{T}_z$ denote the random variable equal to the first time the process attains the value z. Mathematically, this can be expressed as:

Definition 3.9. The first hitting time of a point z in I is defied as:

$$\mathbf{T}_z = \begin{cases} \infty & \text{if } X(t) \neq z, \qquad t \geq 0 \\ \inf\{t \geq 0; X(t) = z\} & \text{otherwise} \end{cases}$$

Karlin and Taylor link the first hitting time of a boundary point with the scale measure, and this relation is used to connect the scale measure with the possibility of reaching a boundary point.

Definition 3.10. *The boundary l is called attracting if*

$$S(l, x_0] < \infty$$

and this criterion applies independently of x_0 in (l, u).

When a boundary point is attractive we have $\mathbb{P}\{\mathbf{T}_{l+} \leq \mathbf{T}_b | X(0) = x_0\} > 0$ for all $x_0 \in (l, b)$. In words this means that there is a chance that the process moves on to the left boundary point before it hits some arbitrary point b to the right of the staring point.
When $S(l, x_0] = \infty$ the boundary point l is called non-attracting. When this is the case, $\mathbb{P}\{\mathbf{T}_{l+} \leq \mathbf{T}_b | X(0) = x_0\} = 0$ for all $x_0 \in (l, b)$.

Karlin and Taylor provide an example of a diffusion with an attractive boundary point where the probability of reaching the attractive boundary in finite time is zero. The question of when a boundary point can be reached in *finite* time is therefore of concern. The functional $\Sigma(l)$ is introduced to answer this.

Definition 3.11. $\Sigma(l) := \lim_{a \downarrow l} \int_a^x S[a, y] dM(y)$

Later, to classify boundary behaviour for a specific diffusion, we are going to need a slightly different expression for $\Sigma(l)$. Hence the proposition.

Proposition 3.2. $\Sigma(l) = \int_l^x M[\eta, x] dS(\eta)$

Proof. From Definition 3.11,

$$\Sigma(l) = \lim_{a \downarrow l} \int_a^x S[a, y] dM(y).$$

Due to continuity and writing out the measure S we get:

$$\int_l^x S(l, y] dM(y) dy = \int_l^x \Big(\int_l^y \mathfrak{s}(\eta) d\eta \Big) \mathfrak{m}(y) dy = \int_l^x \Big(\int_l^y \mathfrak{m}(y) \mathfrak{s}(\eta) d\eta \Big) dy.$$

The reason behind this last step is that for each y fixed, $\mathfrak{m}(y)$ can be considered as a constant with respect to η and can hence be moved inside the integral. This is again the same as:

$$\textstyle\int_l^x \int_l^x \Big(\chi_{\eta\leq y}(\eta)\mathfrak{s}(\eta)\mathfrak{m}(y)d\eta\Big)dy.$$

Consider the measure-space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ where $\mathcal{B}(\mathbb{R})$ is the Borel-σ-algebra on $\mathbb{R}$ and λ is the Lebesgue measure. We know that this measure space is σ-finite. Define $f := \chi_{x\leq y}(x)\mathfrak{s}(x)\mathfrak{m}(y)$. Since $f : \mathbb{R}\times\mathbb{R} \to [0,\infty]$ is continuous and thus $\mathcal{B}^2(\mathbb{R}) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$-measurable on $\mathbb{R}\times\mathbb{R}$, we are therefore in the setup where we can apply *Tonelli's Theorem* to interchange order of integration. The expression above thus becomes (note the change of indicator) $\int_l^x \int_l^x \chi_{y\geq\eta}(y)\mathfrak{m}(y)\mathfrak{s}(\eta)dyd\eta = \int_l^x \Big(\int_\eta^x \mathfrak{m}(y)dy\Big)\mathfrak{s}(\eta)d\eta$.

□

Definition 3.12. *The boundary l is said to be:*

i) attainable if $\Sigma(l) < \infty$
ii) unattainable if $\Sigma(l) = \infty$.

We next introduce:

Definition 3.13. $N(l) := \int_l^x S[\eta, x]dM(\eta)$

This functional roughly measures the time it takes to reach an interior point x in (l, u) starting at the boundary l. With this in mind, we can convince ourselves that if $N(l) = \infty$ the diffusion should not be able to move into the interior of the state space I once the boundary l is reached. It would make sense to call such a boundary point for an exit[32] point and this is indeed the case.

Endpoints are either *entrance, exit, natural* or *regular*. William Feller's boundary classification of the boundary point l is based on $S(l, x], \Sigma(l), M(l, x], N(l)$[33]. A boundary point $y \in \{l, u\}$ is said to be:

(i) *entrance* if $\Sigma(y) = \infty$ and $N(y) < \infty$

(ii) *exit* if $\Sigma(y) < \infty$ and $N(y) = \infty$

(iii) *natural* if $\Sigma(y) = \infty$ and $N(y) = \infty$

(iv) *regular* if $\Sigma(y) < \infty$ and $N(y) < \infty$

3.4 Recovery with unbounded diffusions

Matrices yield examples of linear maps. When considering matrix equations we know how useful the property of invertibility is. In our analysis, to find the shape of the numeraire portfolio we need to solve a differential equation. Since we have formulated this problem

[32] or an absorbing state or for a trap
[33] with appropriate modifications of the functionals for the upper boundary point u.

in terms of a linear operator, a question that should concern us is whether or not this differential operator is invertible. Unfortunately, the appropriate integral operator is not in any sense an inverse of the differential problem, it merely converts a differential equation problem into an integral equation. To find the function we are interested in we still need to solve an equation. However, by finding a function satisfying some conditions we can find the *fundamental solution* associated to the differential operator. This function is often called *Green's function.*

To illustrate, let L be the operator introduces in Appendix A,

$$Ly := a_2(x)y'' + a_1(x)y' + a_0(x)y.$$

Green's function $G(x, \eta)$ is obtained as follows:
Suppose the homogenous equation $Ly = 0$ admits no non-zero solutions fulfilling the boundary conditions. Let $y_1(x)$ $[y_2(x)]$[34] be a solution of $Ly = 0$ satisfying the initial conditions $y_1(l) = 0$ with $y_1'(l) > 0$ $[y_2(u) = 0, y_2'(u) < 0]$. y_1 $[y_2]$ is determined up to a multiplicative constant as a solution of $Ly = 0$ satisfying the lower [upper] boundary condition. In most examples these exist and are easily determined. The functions $y_1(x)$ and $y_2(x)$ are linearly independent since we assumed there was no nonzero solution of $Ly = 0$ satisfying both boundary conditions. We can introduce the *Wronskian*, a function sometimes used to show that a set of solutions is linearly independent. That is,
$W(\eta) = y_1(\eta)y_2'(\eta) - y_2(\eta)y_1'(\eta)$ with the property of being nonzero for $l \leq \eta \leq u$. If we form:

$$G(x, \eta) = \begin{cases} -\frac{y_1(x)y_2(\eta)}{W(\eta)a_2(\eta)} & \text{for } l \leq x \leq \eta \leq u, \\ -\frac{y_1(\eta)y_2(x)}{W(\eta)a_2(\eta)} & \text{for } l \leq \eta \leq x \leq u, \end{cases}$$

the function $w(x) = \int_l^u G(x, \eta)f(\eta)d\eta$ satisfies the boundary conditions. Calculating the derivatives for $w(x)$ we find that $w(x)$ satisfies $Lw(x) = -f(x)$.[18, p.199-202]

The insight we have gained from the argument above will be useful in solving the differential equation we are interested in. With reference to Borodin and Salminen[33], if we let $\mathbf{T}_z$ be the first hitting time of $z \in I$, then for $\alpha > 0$, we have that the non-negative random variable $\mathbf{T}_z$ has the Laplace transformation:

$$\mathrm{E}_x\left[e^{-\alpha \mathbf{T}_z}\right] = \begin{cases} \frac{\psi_\alpha(x)}{\psi_\alpha(z)} & x \leq z \\ \frac{\phi_\alpha(x)}{\phi_\alpha(z)} & x \geq z \end{cases}$$

where $\psi_\alpha(x)$ and $\phi_\alpha(x)$ are continuous solutions of the SL differential equation:

$$\mathcal{G}^e_{x,t}u(x) = b(x)u'(x) + \frac{1}{2}a^2(x)u''(x) - r(x)u(x) = \alpha u(x)^{35} \tag{3.3}$$

[34] Assume for simplicity the following boundary conditions: $y(l) = y(u) = 0$ (other boundary conditions can be handled by similar means[18, p.199]).

[35] since $u = u(x)$ and hence $\frac{\partial}{\partial t}u(x) = 0$

The functions $\psi_\alpha(x)$ and $\phi_\alpha(x)$ can be characterized as the unique, up to a multiplicative constant dependent on α but independent of x, solution of (3.3) by firstly, as above, demanding that $\psi_\alpha(x)$ is increasing in x and $\phi_\alpha(x)$ is decreasing in x, and secondly posing conditions at boundary points.[33] As an example, if l is natural we have:

$$\begin{array}{rl} \psi_\alpha(l+) = 0, & \lim_{x \downarrow l} \frac{\psi'_\alpha(x)}{\mathfrak{s}(x)} = 0 \\ \phi_\alpha(l+) = +\infty, & \lim_{x \downarrow} \frac{\phi'_\alpha(x)}{\mathfrak{s}(x)} = -\infty \end{array}$$

Analogous properties hold at u with ψ and ϕ interchanged. We have different boundary conditions for states with other classifications. These conditions can be found in Borodin and Salminen.[33]
The functions $\psi_\alpha(x)$ and $\phi_\alpha(x)$ are called *fundamental solutions* of the SL equation (3.3), they are linearly independent and all solutions can be expressed as their linear combinations.

Investigating the qualitative nature of the spectrum and obtaining a general spectral expansion is not straightforward. In Theorem 3.1 we learned that it was of importance that the generator was on self-adjoint form. Consider 3.3 with $\alpha = -\lambda \in \mathbb{C}$. From Proposition 3.1 we know that 3.3 can be written as:

$$-\mathcal{G}^e_{x,t} u(x) = -\frac{1}{\mathfrak{m}(x)} \left(\frac{u'(x)}{\mathfrak{s}(x)} \right)' + r(x)u(x) = \lambda u(x) \tag{3.4}$$

(3.4) is on self-adjoint form since it is on SL form (Appendix A). The oscillatory/non-oscillatory classification based on Sturm's theory of oscillations of solutions is of fundamental importance in determining the qualitative nature of the spectrum of the SL operator. *"For a given $\lambda \in \mathbb{R}$,* (3.4) *is oscillatory at an endpoint $y \in \{l, u\}$ if and only if every solution has infinitely many zeros clustering at y. Otherwise it is called non-oscillatory at y."*[32, p.245]
From SL theory we have that entrance, exit and regular boundaries are always non-oscillatory for the associated SL equation (see [32, p.245-247]). If both endpoints are non-oscillatory, then the spectrum is simple, non-negative and purely discrete.
Natural boundaries can be either non-oscillatory or oscillatory with cutoff $\Lambda \geq 0$ [36]. Oscillatory natural boundaries generate some non-empty essential spectrum[37] above the cutoff.

[36] Theorem 3.1 in Linetsky[32, p.246] is about Oscillatory/Non-oscillatory Classification of Boundaries. In this theorem we learn that boundary points are either non-oscillatory for all real $\lambda = -\alpha$ in (3.4) or that there exists a real number $\Lambda \geq 0$ such that the SL equation is oscillatory at a boundary point for all $\lambda > \Lambda$ and non-oscillatory for all $\lambda < \Lambda$.

[37] We say that $\lambda \in \sigma(T)$ is in the essential spectrum of T, $\sigma_e(T)$, if and only if the range of $E\big((\lambda-\epsilon, \lambda+\epsilon)\big)$ is infinite-dimensional for all $\epsilon > 0$. The spectrum of an operator T can be decomposed into two disjoint components, $\sigma(T) = \sigma_e(T) \cup \sigma_d(T)$, where $\sigma_d(T)$ is the discrete spectrum. $\lambda \in \sigma_d(T)$ if and only if λ is an isolated point of $\sigma(T)$ *and* λ is an eigenvalue of finite multiplicity. Therefore the essential spectrum contains the absolute continuous spectrum and some possible complicated singular continuous spectrum.[32, p.232-234]

Linetsky transforms the SL equation to the so-called *Liouville normal form* to determine when a natural boundary point is non-oscillatory or oscillatory with cutoff Λ. The oscillatory/non-oscillatory classification of boundaries of the SL equation remains invariant under the Liouville transform. The SL equation in the Liouville normal form has the shape of the celebrated (*one-dimensional*) *Schrödinger equation.* This transformation is fruitful since that in order to determine the spectral representation explicitly, one needs explicit solutions of the SL equation. If analytical solutions are available for the Schrödinger equation, inverting the Liouville transform yields analytical solutions to the original SL equation. Since the Scrödinger equation is the fundamental equation of quantum mechanics, it has been intensively studied in mathematical physics and we therefore have analytical solutions available for a vast list of specifications.

3.5 CIR

If we let $a(x) = \sigma\sqrt{x}$ and $b(x) = \kappa(\theta - x)$ we get a diffusion known in finance as the Cox Ingersoll Ross (CIR) process. Here $\kappa > 0$, $\theta > 0$ and $\sigma > 0$ are the rate of mean reversion, the longrun level, and volatility respectively. This diffusion is widely used as a model of interest rates, stochastic volatility, and credit spreads. If the Feller condition $2\kappa\theta > \sigma^2$ is satisfied then there exists an unique positive solution of the diffusion.[34]
The non-negativity property of the CIR process has been thought as reasonable when considering interest rates, even though, as recent events have showed, negative interest rates can occur[38].

First, recall Definition 3.7. Since $\frac{2b(y)}{a^2(y)} = \frac{2\kappa\theta}{\sigma^2 y} - \frac{2\kappa}{\sigma^2}$, we arrive at the following expression by calculating the appropriate integral: $\mathfrak{s}(x) = x^{-\frac{2\kappa\theta}{\sigma^2}} e^{\frac{2\kappa}{\sigma^2}x}$.

Second, recall Definition 3.8. Since we already have calculated the speed density we observe that $\mathfrak{m}(x) = \frac{2}{\sigma^2}x^{-1}x^{\frac{2\kappa\theta}{\sigma^2}}e^{-\frac{2\kappa}{\sigma^2}x}$. By defining well-suited constants we can write this as:

$$\mathfrak{s}(x) = x^{-\beta}e^{\vartheta x}, \qquad \mathfrak{m}(x) = \tfrac{2}{\sigma^2}x^{\beta-1}e^{-\vartheta x}$$
$$\text{where } \beta := \tfrac{2\kappa\theta}{\sigma^2}, \vartheta := \tfrac{2\kappa}{\sigma^2}$$

For $\beta > 1$, zero is an inaccessible entrance boundary.[35] In order to ascertain the characteristics of the boundary ∞ we need to estimate $\Sigma(\infty)$ and $N(\infty)$ (see Appendix C).

The fundamental solutions $\psi_\alpha(x), \phi_\alpha(x)$ can be found to be functions of the Kummer and Tricomi confluent hypergeometric functions.[32, p.267-268] The Wronskian is
$w_\alpha = \frac{\Gamma(\beta)}{\Gamma(\frac{\alpha}{\kappa})}\vartheta^{-\beta+1}$. Here Γ is the well-known gamma function. The reason why the Wronskian is important is since an eigenfunction $\varphi_n(x)$ satisfies the SL equation with $\alpha = -\lambda_n$ and also satisfies the appropriate boundary condition at the left endpoint. Hence it must be

[38] http://www.bloombergview.com/quicktake/negative-interest-rates

equal to $\psi_{-\lambda_n}(x)$ up to a nonzero constant multiple. But $\varphi_n(x)$ also satisfies the appropriate boundary condition at the right endpoint and must therefore be equal to $\phi_{-\lambda_n}(x)$ up to a non-zero multiple. Thus, for $\alpha = -\lambda_n$, $\psi_{-\lambda_n}(x)$ and $\phi_{-\lambda_n}(x)$ must be linearly dependent and hence their Wronskian must vanish for $\alpha = -\lambda_n$. Conversely, let $\alpha = -\lambda_n$ be a zero of the Wronskian. Then the fundamental solutions $\phi_{-\lambda_n}(x)$ and $\psi_{-\lambda_n}(x)$ are linearly dependent, and hence $\psi_{-\lambda_n}(x)$ is a solution satisfying *both* boundary conditions. Therefore $\psi_{-\lambda_n}(x)$ is a *non-normalized* eigenfunction corresponding to $-\lambda_n$. We have that$\{-\lambda_n\}_{n\in\mathbb{N}}$ are zeros of the Wronskian. This approach can therefore be used to determine the eigenvalues when we know that the spectrum is purely discrete. When we have performed the Liouville transform we use solutions of the Scrödinger equation to find the properties of the spectrum. We now have a method of solving the SL equation and obtaining an eigen expansion.

For the CIR process, the zeros of the Wronskian are $\alpha = -\lambda_n$, where $\lambda_n = \kappa n, n = 0, 1, 2, ...$[39]. The spectrum is purely discrete with the eigenvalues $\lambda = \kappa n$. The normalized associated eigenfunctions are $\varphi_n(x) = \sqrt{\frac{n!\kappa}{\Gamma(\beta+n)}}\vartheta^{\frac{\beta-1}{2}}\Lambda_n^{(\beta-1)}(\vartheta x)$, where $\Lambda_n^{(\alpha)}(x)$ are the generalized Laguerre polynomials. The first eigenfunction expansion gives the stationary density of the CIR process, namely the gamma density.[40]

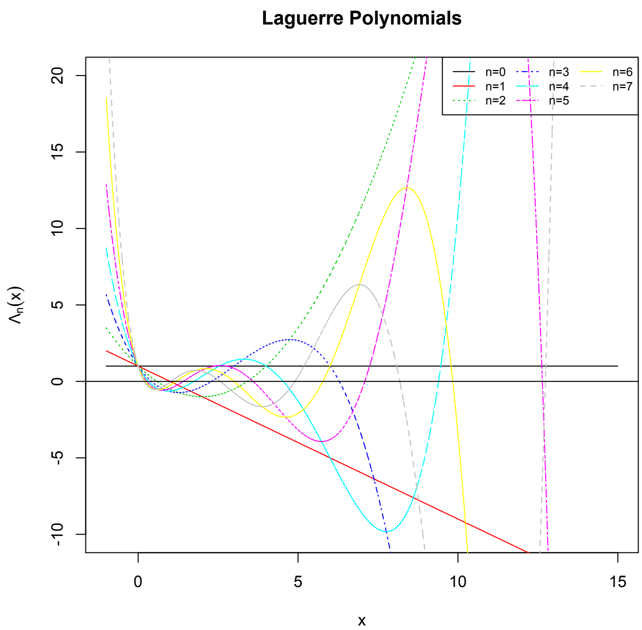

Figure 2: The first Normalized Laguerre Polynomials

This is useful. The numeraire portfolio is strictly positive. By inspecting the generalized Laguerre polynomials we understand that only the first polynomial is non-negative for all

[39]For notational convenience we here label the eigenvalues starting from zero (see [32, p.268]).

[40] Appendix B. For more details see [32, p.267-269].

values of $x \geq 0$. Therefore, as SL theory predicts, the first eigenfunction is the only one not changing signs. Therefore, Long's numeraire must consist of the first eigenfunction. Since $\lambda_0 = 0$ we have $L(t, x) = e^{0 \cdot t}\varphi_0(x) = \varphi_0(x)$.

3.6 Illustrating graph of result

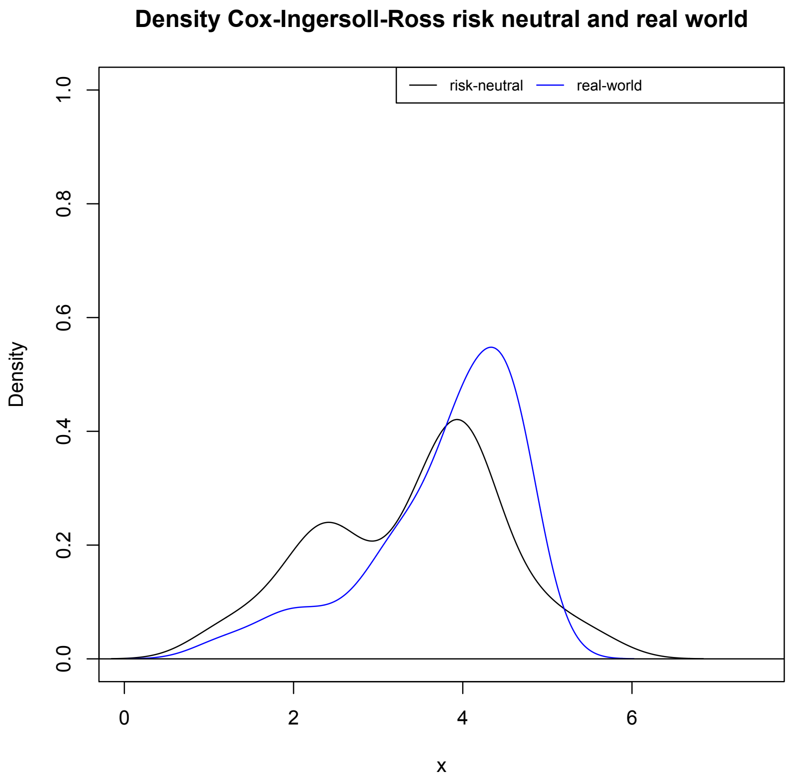

Figure 3: Driving diffusion adjusted by market price of risk

The graph above is just ment as an illustration. Here I have used $\kappa = 0.7, \kappa\theta = 3.2 \Rightarrow \theta \approx 4.5714$ and $\sigma = 0.5$. The parameters of the numeraire portfolio follow by using the connection established in Appendix B. We see that the natural density, as expected, is shifted to the right. I am not satisfied with the result the density in the risk-neutral world has a fatter right tail.

4 Concluding remarks

Separating the risk-neutral density extracted from option prices into the predicted natural density with corresponding market price of risk is of interest due to the rich source of information for gauging market sentiment it might provide us with. We can possibly use the market's future distribution of returns much as we use forward rates as forecasts of future spot rates. For a time, economists have been aware of the challenges we face when using historical time series of returns to estimate downturns in the future. History may fool us to believe that the future is more certain than what is really the case.

Firstly, the purpose of this thesis has been to understand Carr and Yu's proposed method for recovery. By carefully inspecting and building up piece by piece the results stated in Carr and Yu's paper, we have gained valuable insight.[13] This insight has been crucial when we have tried to extend their result into the more familiar framework of mathematical finance. Using unbounded stochastic processes as driving forces of the economy will allow us to use the vehicles already at hand in mathematical finance.

Properties with the Sturm-Liouville eigenvalue problem is what Carr and Yu use to find an explicit expression for Long's numeraire portfolio and thus finding the market price of risk, and hence recover the real-world density from the risk-neutral density extracted from option prices. When the driving diffusion is *bounded* and we impose appropriate boundary conditions, we have from Remark 2 that the SL problem is *regular*. From Appendix A we have that a regular SL problem obtains an eigenfunction expansion. Therefore, boundedness of the one-dimensional diffusion used as the driving process of the economy is a *sufficient* condition for obtaining and eigenfunction expansion, allowing us to recover.

As we learned from Remark 2, boundedness of the diffusion is not a *necessary* condition for having a discrete spectrum of the SL operator. *"...the spectrum of a singular* (*SL*) *problem can be discrete, continuous, or mixed, and further analysis is needed to determine the nature of the spectrum in each case."*[30, p.190] From Equation (3.2) we have that when the spectrum of the extended generator is purely discrete, the spectral representation in Theorem 3.2 simplifies to an eigenfunction expansion.
The second purpose of this thesis has been to understand when unbounded diffusions fit into the framework developed by Carr and Yu, i.e. when the SL operator of an unbounded one-dimensional diffusion process has a discrete spectrum.

Due to the generality of our aim, we have been led onto a technical path. Luckily we have come across brilliant textbooks concerned with understanding boundary behaviour of diffusion processes and spectral decomposition of Sturm-Liouville operators. Karlin and Taylor and Vadim Linetsky have been indispensable.[18] [32]
When we understood that our differential equation of concern already has been solved for a vast list of diffusion-specifications due to its importance in mathematics and mathematical physics, explicit solutions for Long's numeraire portolio could be found. But our approach has clear limitations. If we model the driving process of the economy by Brownian motion alone, since the spectral decomposition is continuous (see [18, p.337]), we can not recover the real-world density with our approach.

It is worth mentioning that finding the eigenfunction expansion for an unbounded diffusion process does not need to be as complicated as for the CIR process. If we were to model the economy by an Ornstein-Uhlenbeck process, Karlin and Taylor classify the boundary points and derive the eigenfunction expansion without turning to the Liouville transformation.[18, p.237-238] [18, p.322-323]

In Appendix A, we note that the boundary values are an important part of the SL eigenvalue problem. One could argue that if there is no general method of obtaining the boundary conditions for the numeraire portfolio, the implementation of the recovery theorem in Carr and Yu's paper could be difficult in practice.
When working with unbounded diffusions, we had to analyse boundary behaviour more carefully, saving us from the temptation of imposing ad-hoc boundary conditions.

A fruitful investigation would be to apply our findings on real data. I had hoped, though a bit optimistic, to have enough time to start out with this exercise. It would be especially interesting to extract predicted densities with matching market price of risk for two dates close in time, but with some adverse event in between. It would be alluring to compare the findings for these two dates and check out if our recovered data would suggest a change of perception of the future. If so, recovery could provide us with insight into the market's future perception more adaptive than historical data.

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A Appendix A: Regular Sturm-Liouville(SL) Problem

Due to [36].
In many situations, we are interested in solving a second-order ordinary differential equation. For example, we might want to solve the equation

$$a_2(x)y'' + a_1(x)y' + a_0 y = f(x)$$

subject to boundary conditions. We have stressed the concept of an operator in this text, and thus we know that we can write the equation above on operator form by defining the differential operator

$$L := a_2(x)\tfrac{d^2}{dx^2} + a_1(x)\tfrac{d}{dx} + a_0(x)$$

It turns out that *any* linear second-order operator can be turned into an operator that possesses just the right properties, i.e. *self-adjointness*. The resulting operator is referred to as the *Sturm-Liouville*(*SL*) operator. We define the SL operator as:

$$\mathcal{L} = \frac{d}{dx}p(x)\frac{d}{dx} + q(x) \tag{A.1}$$

The SL eigenvalue problem is given by the differential equation

$$\mathcal{L}y = -\lambda\rho y,$$

or

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda\rho(x)y = 0 \tag{A.2}$$

for $x \in (l, u)$. The functions $p(x), p'(x), q(x)$ and $\rho(x)$ are assumed to be continuous on (l, u) and $p(x) > 0, \rho(x) > 0$ on $[l, u]$. If the interval is finite and these assumptions on the coefficients are true on $[l, u]$, the the problem is said to be *regular*. Otherwise, it is called *singular*.
We also need to impose the set of homogenous boundary conditions

$$\alpha_1 y(l) + \beta_1 y'(l) = 0 \tag{A.3}$$

$$\alpha_2 y(u) + \beta_2 y'(u) = 0 \tag{A.4}$$

The $\alpha' s$ and $\beta' s$ are constant.

We are interested in a general linear second-order ODE on the form

$$y'' + g(x)y' + h(x)y + \lambda\rho(x)y = 0$$

Define the integrating factor

$$z(x) := \exp\{\int_l^x g(s)ds\}.$$

We note that z is positive and $z' = zg$. If we multiply the ODE above by z we get:

$$zy'' + zgy' + zhy + \lambda z\rho y = 0 \Rightarrow$$
$$(zy')' + zhy + \lambda z\rho y = 0$$

We observe that this last expression is on the Sturm-Liouville form.

There are several important properties for the regular Sturm-Liouville eigenvalue problem. Here we will merely list some of the important facts useful to us.

1. The eigenvalues are real, countable, ordered and there is a smallest eigenvalue. Thus, we can write them as $\lambda_1 < \lambda_2 < ....$ However, there is no largest eigenvalue, and $n \to \infty, \lambda_n \to \infty$

2. For each eigenvalue λ_n there exists an eigenfunction ϕ_n with $n-1$ zeros on (l, u). $\phi_n(x)$ is unique up to a normalising constant, and $\phi_n(x)$ is called the nth fundamental solution. In particular, the first fundamental solution has no zeros in (l, u) and can always taken to be positive.

3. Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function $\rho(x)$. Defining the inner product of $f(x)$ and $g(x)$ as:

   $$\langle f, g\rangle = \int_l^u f(x)g(x)\rho(x)dx$$

   then the orthogonality of the eigenfunctions can be written in the form:

   $$\langle \phi_n, \phi_m\rangle = \langle \phi_n, \phi_n\rangle \delta_{n,m}, n, m = 1, 2, ...$$

   Here $\delta_{n,m}$ is the Kronecker delta, defined as:

   $$\delta_{n,m} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

4. The set of eigenfunctions is complete, i.e. any $f \in L^2_\rho[l, u]$ can be represented by a general Fourier series expansion of the eigenfunctions,

   $$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

   where

   $$c_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}$$

B Appendix B: Stationary distribution for the CIR process

The following argument is due to professor Fred Espen Benth.[27]
Let $r(t)$ be an Itô diffusion. With reference to Proposition 2.2 we have

$$\mathrm{E}^{\mathbb{P}}\Big[e^{zr(T)}|\mathcal{F}_t\Big] = F(t,r) \tag{B.1}$$

for some function $F(t,x)$ where in most cases F is highly regular.
Define

$$M(t) := \mathrm{E}^{\mathbb{P}}\Big[e^{zr(T)}|\mathcal{F}_t\Big].$$

Suppose $t \geq u$. Since

$$\mathrm{E}^{\mathbb{P}}\Big[M(t)|\mathcal{F}_u\Big] = \mathrm{E}^{\mathbb{P}}\Big[\mathrm{E}^{\mathbb{P}}[e^{zr(T)}|\mathcal{F}_t]|\mathcal{F}_u\Big] = \mathrm{E}^{\mathbb{P}}\Big[e^{zr(T)}|\mathcal{F}_u\Big] = M(u)$$

we have that, if $\mathrm{E}^{\mathbb{P}}\Big[|M(t)|\Big] < \infty$ for all $t \in [0,T]$, $M(t)$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ under $\mathbb{P}$. Knowing this will be useful for us determining the distribution. This is due to the fact that by definition, $M(t)$ is the *moment-generating function*(mgf). *"If the mgf exist for z in an open interval containing zero, it uniquely determines the probability distribution."*[37]
If we apply Itô's formula to $F(t,r(t))$ we get:

$$dF = \tfrac{\partial F}{\partial t}dt + \tfrac{\partial F}{\partial x}dr + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial^2 x}(dr)^2.$$

Further, assuming $r(t)$ follows the CIR dynamic the equation above becomes:

$$dF = \Big(\tfrac{\partial F}{\partial t} + \tfrac{\partial F}{\partial x}\kappa(\theta - r(t)) + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial^2 x}\sigma^2 r(t)\Big)dt + \tfrac{\partial F}{\partial x}\sigma\sqrt{r(t)}dB_t^{\mathbb{P}}.$$

From Equation (B.1) and by the definition of $M(t)$, we have $M(t) = F(t,r(t))$ and since $M(t)$ is a martingale, we know from The Martingale Representation Theorem, Theorem 2.2, that $M(t)$ can be written uniquely as a pure Itô integral and hence the dt-terms in the expression above must equate to zero. This condition must hold for *any* value of $r(t) \geq 0$, hence:

$$\tfrac{\partial F(t,x)}{\partial t} + \tfrac{\partial F(t,x)}{\partial x}\kappa(\theta - x) + \tfrac{1}{2}\tfrac{\partial^2 F(t,x)}{\partial^2 x}\sigma^2 x = 0.$$

By studying this PDE carefully, guessing that the solution $F(t,x)$ is on the form $F(t,x) = e^{a(t,T)x+b(t,T)}$ does not seem to unreasonable. The reader familiar with *affine* term structure models know that the CIR-process is affine, i.e. that the solution of the Zero Coupon Bond(ZCB) prices are on the form $P(t,T) = e^{a(t,T)-B(t,T)r(t)}$, where $A(t,T), B(t,T)$ are deterministic functions.[24] By an obvious calculation, assuming the functions $t \to a(t,T), t \to b(t,T)$ are differentiable, we get:

$$F(t,x)a'(t,T)x + F(t,x)b'(t,T) + F(t,x)a(t,T)\kappa(\theta - x) + \tfrac{1}{2}F(t,T)a^2(t,T)\sigma^2 x = 0$$

Again, since this must be true for *any* value of x and since $F > 0$ we get the following system of differential equations, or a system of Ricatti equations.

$$a'(t,T) - \kappa a(t,T) + \frac{1}{2}a^2(t,T)\sigma^2 = 0 \tag{B.2}$$

$$b'(t,T) + \kappa\theta a^2(t,T) = 0 \tag{B.3}$$

The boundary conditions are:

$$a(T,T) = z, b(t,T) = 0$$

Recalling the explicit expression for the solutions of the ZCB prices under the CIR model and taking the appropriate boundary conditions into considerations, a *well-certified guess* is that

$$a(t,T) = \tfrac{e^{c(T-t)}}{\alpha + \beta e^{c(T-t)}} \text{ for some constants } c, \alpha, \beta.$$

Firstly,

$$a(T,T) = z \Rightarrow \frac{1}{\alpha + \beta} = z \tag{B.4}$$

Secondly

$$a'(t,T) = -ca(t,T) - (\frac{e^{c(T-t)}}{(\alpha + \beta e^{c(T-t)})^2})(\beta(-c)e^{c(T-t)}) = -ca(t,T) + c\beta a^2(t,T) \tag{B.5}$$

Comparing (B.5) with (B.2) we see that $c = -\kappa$. Further, we must have $\kappa\beta = \frac{1}{2}\sigma^2 \Rightarrow \beta = \frac{\sigma^2}{2\kappa}$. Using this in (B.4), we must have

$$\alpha = \tfrac{1}{z} - \tfrac{\sigma^2}{2\kappa}.$$

Combining all of this we get that

$$a(t,T) = \tfrac{e^{-\kappa(T-t)}}{\frac{1}{z} - \frac{\sigma^2}{2\kappa} + \frac{\sigma^2}{2\kappa}e^{-\kappa(T-t)}}$$

Next, from (B.3)

$$b'(t,T) = -\kappa\theta a(t,T).$$

Writing this on integral form we get

$$b(T,T) - b(t,T) = \textstyle\int_t^T b'(s,T)ds.$$

We use that $b(T,T) = 0$ and thus get:

$$b(t,T) = -\textstyle\int_t^T b'(s,T)ds = \int_t^T \kappa\theta a(s,T)ds.$$

We multiply $a(s,T)$ by $\frac{z}{z}$ and define

$$u(s) := 1 - \tfrac{z\sigma^2}{2\kappa} + \tfrac{z\sigma^2}{2\kappa}e^{-\kappa(T-s)}$$

By doing this we note

$$b(t,T) = \kappa\theta \int_t^T \tfrac{ze^{-\kappa(T-s)}}{u(s)}ds.$$

Further $du = \frac{z\sigma^2}{2\kappa}e^{-\kappa(T-s)}\kappa ds \Rightarrow$

$$b(t,T) = \kappa\theta \int\limits_{u(t)}^{u(T)} \tfrac{1}{u}\tfrac{2}{\sigma^2}du = \tfrac{2\kappa\theta}{\sigma^2}\Big(\ln(1) - \ln(1 - \tfrac{z\sigma^2}{2\kappa} + \tfrac{z\sigma^2}{2\kappa}e^{-\kappa(T-t)})\Big) =$$

$$-\tfrac{2\kappa\theta}{\sigma^2}\ln(1 - \tfrac{z\sigma^2}{2\kappa} + \tfrac{z\sigma^2}{2\kappa}e^{-\kappa(T-t)}).$$

There may or may not exist a stationary measure approached by the probability distribution of $r(T)$ as $T \to \infty$. If this limit exists, then the process is *strongly recurrent*(*or positive ergodic*), meaning that probability mass cannot escape to the boundaries. The CIR-process is ergodic, meaning there exists a unique stationary distribution to which the process converges as $T \to \infty$.[18, p.237] Therefore we investigate the mgf when $T \to \infty$. We know that if g, f is continuous, then $h := g \circ f$ is continuous. Using this combined with the observation that $\lim_{T\to\infty} a(t,T) = 0$ for $\kappa > 0$ and

$$\lim_{T\to\infty} \ln(1 - \tfrac{z\sigma^2}{2\kappa} + \tfrac{z\sigma^2}{2\kappa}e^{-\kappa(T-t)}) = \ln(1 - \tfrac{z\sigma^2}{2\kappa})$$

we get that

$$\lim_{T\to\infty} \mathrm{E}^{\mathbb{P}}\Big[e^{zr(T)}|r(t)\Big] = e^{\lim_{T\to\infty} a(t,T)r(t) + \lim_{T\to\infty} b(t,T)} =$$

$$e^{-\frac{2\kappa\theta}{\sigma^2}\ln(1-\frac{z\sigma^2}{2\kappa})} = (1 - \tfrac{z\sigma^2}{2\kappa})^{-\frac{2\kappa\theta}{\sigma^2}} = \Big(\tfrac{\frac{2\kappa}{\sigma^2}}{\frac{2\kappa}{\sigma^2}-z}\Big)^{\frac{2\kappa\theta}{\sigma^2}}.$$

This is the mgf of the Γ-distribution, where $\gamma = \frac{2\kappa}{\sigma^2}$ is the scale and $\delta = \frac{2\kappa\theta}{\sigma^2}$ is the shape.[37]

C Appendix C: Classification of ∞ for CIR

As mentioned, we need to estimate $\Sigma(\infty)$ and $N(\infty)$. From definition 3.11, making the appropriate adjustment for the right boundary point u we have:

$$\Sigma(\infty) = \int_d^\infty \Big(\int_x^y dM(\eta)\Big)dS(y) = \int_d^\infty \Big(\int_d^y \frac{2}{\sigma^2}\eta^{\beta-1}e^{-\vartheta\eta}d\eta\Big)y^{-\beta}e^{\vartheta y}dy \tag{C.1}$$

The inner integral is not straightforward to calculate. If d is taken to be zero, this integral is related to the *lower incomplete gamma function*, $\gamma(\beta, x) := \int_0^x \eta^{\beta-1}e^{-\eta}d\eta$. Luckily our task is only to find out if $\Sigma(\infty)$ is finite or not. From Karlin and Taylor we have learned that we can suppress the dependence on d without ambiguity. Therefore we can safely assume $d = 1$.

In our discussion, we assume $\beta > 1$ so that we know from Gikhman that there exist an unique solution and from Feller that 0 is an inaccessible entrance boundary.[34][35] Also, the diffusion lives on $(0, \infty)$ so $\eta > 0$. Because of this we know

$$\eta^{\beta-1}e^{-\vartheta\eta} \geq \eta^0 e^{-\vartheta\eta} \Rightarrow \int_1^y \eta^{\beta-1}e^{-\vartheta\eta}d\eta \geq \int_1^y e^{-\vartheta\eta}d\eta = \tfrac{1}{\vartheta}(e^{-\vartheta} - e^{-\vartheta y}) \Rightarrow$$
$$\int_1^y \eta^{\beta-1}e^{-\vartheta\eta}d\eta y^{-\beta}e^{\vartheta y} \geq \tfrac{1}{\vartheta}(e^{-\vartheta} - e^{-\vartheta y})y^{-\beta}e^{\vartheta y} = \tfrac{1}{\vartheta}(y^{-\beta}e^{-\vartheta+\vartheta y} - y^{-\beta}e^{-\vartheta y+\vartheta y}) =$$

$$\frac{1}{\vartheta}(y^{-\beta}e^{-\vartheta+\vartheta y} - y^{-\beta}) \tag{C.2}$$

In mathematics, the expression $\infty - \infty$ is not well defined. Therefore we must make sure that when we split up the expression above and perform the outer integral, we subtract something finite.

$$\int_1^\infty y^{-\beta}dy = \lim_{A\to\infty} \tfrac{1}{-\beta+1}[y^{-\beta+1}]_1^A =$$
$$\lim_{A\to\infty} \tfrac{1}{-\beta+1}A^{-\beta+1} - \tfrac{1}{-\beta+1} = \tfrac{-1}{-\beta+1} < \infty \text{ since } \beta > 1.$$

We note that our chosen approach will lead to difficulties when $\beta = 1$, i.e. we get

$$\int_1^\infty y^{-1}dy = [\ln y]_1^\infty = \infty$$

and possible gives us the problem of evaluating the ill-defined $\infty - \infty$.
When $\beta > 1$, we get that we subtract a finite number, and hence to determine whether $\Sigma(\infty)$ is finite or not, we have from (C.1) and (C.2) that we have to estimate

$$\int_1^\infty \tfrac{1}{\vartheta}y^{-\beta}e^{-\vartheta+\vartheta y}dy.$$

We note $e^{-\vartheta}$ is just a positive constant and is therefore not of importance when estimating whether or not $\Sigma(\infty)$ is finite.
We know

$$e^{\vartheta y} = \sum_{n=0}^\infty \tfrac{(\vartheta y)^n}{n!}$$

Since $\vartheta > 0$ and $y > 0$ all the elements in this sum are positive. Hence

$$e^{\vartheta y} = \sum_{n=0}^\infty \tfrac{(\vartheta y)^n}{n!} \geq \sum_{n=0}^N \tfrac{(\vartheta y)^n}{n!} \text{ for } \textit{any } N \in \mathbb{N}.$$

Due to this we have that

$$y^{-\beta}e^{\vartheta y} \geq y^{-\beta}\sum_{n=0}^N \tfrac{(\vartheta y)^n}{n!} = \sum_{n=0}^N \tfrac{\vartheta^n y^{-\beta+n}}{n!} \Rightarrow$$
$$\int_1^\infty y^{-\beta}e^{\vartheta y}dy \geq \int_1^\infty y^{-\beta}\sum_{n=0}^N \tfrac{(\vartheta y)^n}{n!}dy = \sum_{n=0}^N \int_1^\infty \tfrac{\vartheta^n y^{-\beta+n}}{n!}dy.$$

We note that, for n fixed

$$\int_1^\infty \tfrac{\vartheta^n}{n!}y^{-\beta+n}dy = \begin{cases} \lim_{A\to\infty} \tfrac{1}{-\beta+n+1}[\tfrac{\vartheta^n}{n!}y^{-\beta+n+1}]_1^A & \text{if } -\beta+n \neq -1 \\ \lim_{A\to\infty}[\tfrac{\vartheta^n}{n!}\ln(y)]_1^A & \text{if } -\beta+n = -1 \end{cases}$$

Since

$$\lim_{A\to\infty} A^{-\beta+n+1} = \begin{cases} 0 & \text{if } -\beta+n+1 < 0 \\ \infty & \text{if } -\beta+n+1 > 0 \end{cases}$$

we therefore have

$$\int_1^\infty \frac{\vartheta^n}{n!} y^{-\beta+n} dy = \begin{cases} \frac{-1}{-\beta+n+1}\frac{\vartheta^n}{n!} & \text{if } -\beta+n+1 < 0 \\ \infty & \text{if } -\beta+n+1 \geq 0 \end{cases}$$

Because β is a finite number, we can always choose n such that $-\beta+n+1 \geq 0$. Therefore, $\sum_{n=0}^{N} \int_1^\infty \frac{\vartheta^n y^{-\beta+n}}{n!} dy$ consists of at least one non-finite term and the sum is therefore not finite. Due to the direction of the inequality we deduce that $\Sigma(\infty) = \infty$.

Next, we need to estimate $N(\infty)$. From Definition 3.13 and with the appropriate modifications for the right boundary point, we have

$$N(\infty) = \int_d^\infty \Big(\int_d^\eta dS(y) \Big) dM(\eta) = \int_d^\infty \Big(\int_d^\eta y^{-\beta} e^{\vartheta y} dy \Big) \tfrac{2}{\sigma^2} \eta^{\beta-1} e^{-\vartheta\eta} d\eta =$$
$$\int_d^\infty \Big(\int_d^\infty \chi_{y\leq\eta}(y) y^{-\beta} e^{\vartheta y} dy \Big) \tfrac{2}{\sigma^2} \eta^{\beta-1} e^{-\vartheta\eta} d\eta.$$

Due to the same argument as in Proposition 3.2, using Tonelli's theorem the expression above becomes: $\frac{2}{\sigma^2} \int_d^\infty \Big(\int_d^\infty \chi_{\eta\geq y}(\eta) y^{-\beta} e^{\vartheta y} \eta^{\beta-1} e^{-\vartheta\eta} d\eta \Big) dy$. We have assumed $\beta > 1$. This means that the expression $\eta^{\beta-1}$ is nondecreasing in η. Also, in the inner integral we have that $\eta \geq y$. Therefore we know that:

$$\tfrac{2}{\sigma^2} \int_d^\infty \Big(\int_d^\infty \chi_{\eta\geq y}(\eta) y^{-\beta} e^{\vartheta y} \eta^{\beta-1} e^{-\vartheta\eta} d\eta \Big) dy \geq$$
$$\tfrac{2}{\sigma^2} \int_d^\infty \Big(\int_d^\infty \chi_{\eta\geq y}(\eta) y^{-\beta} e^{\vartheta y} y^{\beta-1} e^{-\vartheta\eta} d\eta \Big) dy =$$
$$\tfrac{2}{\sigma^2} \int_d^\infty \Big(\int_y^\infty y^{-\beta+\beta-1} e^{\vartheta y} e^{-\vartheta\eta} d\eta \Big) dy =$$
$$\tfrac{2}{\sigma^2} \int_d^\infty y^{-1} e^{\vartheta y} \Big(\tfrac{1}{\vartheta} e^{-\vartheta y} \Big) dy = \tfrac{2}{\sigma^2\vartheta} \int_d^\infty y^{-1} dy = \tfrac{2}{\sigma^2\vartheta} \Big[\ln(y) \Big]_d^\infty = \infty$$

Since $\Sigma(\infty) = N(\infty) = \infty$, we know by conferring with Feller's boundary conditions in section 3.3, that we are in situation (*iii*) and ∞ is a *natural* boundary point for the CIR-model when $\beta > 1$.